MECO 6345, Fall 2004

MIDTERM EXAM - see answers below

Instructions: You have two hours and forty five minutes to answer the following four questions. Start working on the questions that look most familiar to you, as some questions may be more difficult than others. Each question has a 1/4 weight in your midterm grade. There will be no questions asked during the exam: if you think that something is wrong with a question, write down an assumption that enables you to answer it. Write clearly and legibly, otherwise you will receive reduced credit for your answer. Read the questions carefully and make sure that you answer all questions. Please mind your own work.

Have some fun!

1. An industry consists of n firms: all firms produce at constant marginal cost $c > 0$ and face a linear inverse demand curve $P(Q) = a - Q$ (you may assume $a > c$). Firm 1 is a leader in this market – it chooses quantity q_1^* to maximize its profit. Firms $2 - n$ learn about firm 1's quantity choice and decide on their optimal output levels q_{2-n}^*.

a. Suppose that firm 1 chose quantity q_1. For each of the firms $2 - n$, write down their optimal output choices as functions of q_1. To do so, express each firm's profit as a function of its output and the output of its opponents and maximize it with respect to its quantity choice. Do you need to check the second order conditions?

b. Write down firm 1's profit as a function of its own output. To do so, assume that firms $2 - n$ choose their output according to your computations in (a).

c. Compute the optimal quantity choices of firms $1 - n$ and their profits.

d. What happens with the market price as the number of firms increases without bound? How does this price compare with the competitive price?

2. Consider the output maximization problem (the dual of the cost minimization problem):

$$\max_{x \geq 0} f(x)$$
$$s.t.\ w \cdot x \leq E,$$

in which $w \in \mathbb{R}^n_{++}$ is a vector of factor prices and $E > 0$ is a given level of costs. Suppose that the production function $f(\cdot)$ is strictly quasi-concave, strictly increasing and differentiable. Denote the solution by $x^*(w, E)$ and let $F(w, E)$ be the value function (indirect output function): $F(w, E) = f(x^*(w, E))$.

a. Prove that

$$x_i^*(w, E) = -\frac{\partial F(w, E)/\partial w_i}{\partial F(w, E)/\partial E}, \forall i \in \overline{1, n}.$$

Hint: use the Envelope Theorem.

b. Suppose that $F(w_1, w_2, E) = \frac{E}{2\sqrt{w_1 w_2}}$. Find the associated cost function and the conditional factor demands. Hint: what is the relationship between $F(w_1, w_2, c(w_1, w_2, y))$ and y?

3. Guy Noir intends to insure his house and only property, which is built on the southern slopes of Mount St. Helens. The house is worth $w > 0$. With probability $p \in (0, 1)$, the nearby volcano will erupt and will cause damage to the house in the amount $a \leq w$; with probability $1 - p$ there is no loss. An insurance company offers Guy a policy against this loss with a deductible provision: the insurance company

pays off all but d of the loss if the volcano erupts and charges a premium $\pi(d)$. Guy's preferences are characterized by a strictly increasing, strictly concave and differentiable van Neumann-Morgenstern utility function $u(\cdot)$.

a. Find the function $\pi(d)$ that relates the premium to the amount of the deductible when the insurance company has zero expected profit.

b. Show that, when offered a menu of policies that specifies different values for the deductible d, Guy will prefer the policy that has a zero deductible. That is, if the insurance company offers the menu of policies $\pi(d)$ that you computed at point (a), show that Guy will always prefer the policy $\pi(0)$.

4. Consider the indirect utility function

$$v(p_1, p_2, y) = \frac{2^{2/3} y}{3 p_1^{1/3} p_2^{2/3}}.$$

a. Find the Marshallian demand functions.

b. Find the Hicksian (compensated) demand functions. Hint: what is the relationship between $v(p_1, p_2, e(p_1, p_2, u))$ and u?

c. Find a direct utility function that represents these preferences.

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MIDTERM EXAM – suggested answers

1. Firms $2-n$ choose outputs to satisfy

$$q_i(q_1) = \underset{q}{\operatorname{argmax}}(a - q - \sum_{j \neq i, j \neq 1} q_j - q_1 - c)q;$$

the objective is concave, so at an interior optimum $q_i(q_1) = \frac{a-c-q_1}{n}$. It follows that the leader's optimal quantity choice is

$$q_1^* = \underset{q}{\operatorname{argmax}} \left(\frac{a-c-q}{n} \right) q;$$

therefore, $q_1^* = \frac{a-c}{2}$ and $q_j^* = \frac{a-c}{2n}$ for all $j \in \overline{2,n}$, and thus $P_n^* = c + \frac{a-c}{2n}$. Furthermore, firm 1's profit is $\frac{(a-c)^2}{4n}$, while the profit of firm $j \in \overline{2,n}$ is $\frac{(a-c)^2}{4n^2}$. As $n \to \infty$, $P_n^* \to c$.

2 a. The Lagrangian is $\mathcal{L}(x, \lambda; w, E) = f(x) - \lambda(w \cdot x - E)$; by the Envelope Thm. $\partial F(w,E)/\partial w_i = \partial \mathcal{L}(x^*, \lambda^*; w, E)/\partial w_i = -\lambda^* x_i^*$. Similarly, $\partial F(w,E)/\partial E = \partial \mathcal{L}(x^*, \lambda^*; w, E)/\partial E = \lambda^*$. The desired result follows.

b. Note $F(w_1, w_2, c(w_1, w_2, y)) = y$; thus, $c(w_1, w_2, y) = 2\sqrt{w_1 w_2} y$. Apply Shepard's Lemma to find $x_1(w, y) = \sqrt{\frac{w_2}{w_1}} y$ and $x_2(w, y) = \sqrt{\frac{w_1}{w_2}} y$.

3. a. The insurer's expected profit is $(1-p)\pi - p(a - d + \pi)$. The breaking-even condition implies $\pi(d) = p(a-d)$.

b. Guy's problem is to choose d so that

$$d = \underset{s \geq 0}{\operatorname{argmax}}\, pu(w - s - \pi(s)) + (1-p)u(w - \pi(s));$$

the first order condition requires that d satisfy

$$p\left(-\pi'(d) - 1\right) u'(w - d - \pi(d)) - (1-p)\pi'(d) u'(w - \pi(d)) \leq 0,$$

with equality for an interior solution. Noting that $\pi'(d) = -p$, FOC implies that $u'(w - d - \pi(d)) - u'(w - \pi(d)) \geq 0$. If $d > 0$, $u'(w - d - \pi(d)) > u'(w - \pi(d))$ (since u is concave), which implies that the FOC cannot be satisfied at an interior solution. It follows that at the optimum $d = 0$.

4. a. Apply Roy's identity: $x_1^*(p, y) = y/(3p_1)$ and $x_2^*(p, y) = 2y/(3p_2)$.

b. Note $v(p_1, p_2, e(p_1, p_2, u)) = u$, so that $e(p_1, p_2, u) = 3up_1^{1/3}p_2^{2/3}/2^{2/3}$. Hicksian demands are thus $x_1^h(p, u) = \left(\frac{p_2}{2p_1}\right)^{2/3} u$ and $x_2^h(p, u) = \left(\frac{2p_1}{p_2}\right)^{1/3} u$.

c. (harder way) Let $y = 1$ and solve

$$\begin{aligned} &\min_{p_1, p_2} (\ln K - 1/3 \ln p_1 - 2/3 \ln p_2) \\ &s.t.\ p_1 x_1 + p_2 x_2 = 1. \end{aligned}$$

First order conditions are

$$\begin{aligned} -\frac{1}{3p_1} &= \lambda x_1 \\ -\frac{2}{3p_2} &= \lambda x_2 \end{aligned}$$

and it follows that $\frac{x_1}{x_2} = \frac{p_2}{2p_1}$, so that $p_1 = 1/(3x_1)$ and $p_2 = 2/(3x_2)$. Plug into $v(p, 1)$ to find $u(x_1, x_2) = x_1^{1/3}x_2^{2/3}$.

c. (easier) Observe that $v(p_1, p_2, y) = u(x_1^*, x_2^*) = (y/3p_1)^{1/3}(2y/3p_2)^{2/3} = x_1^{1/3}x_2^{2/3}$.