MECO 6345, Fall 2004

FINAL EXAM

Instructions: You have two hours and forty five minutes to answer the following four questions. Start working on the questions that look most familiar to you, as some questions may be more difficult than others. Each question has a 1/4 weight in your grade. There will be no questions asked during the exam: if you think that something is wrong with a question, write down an assumption that fixes the problem. Write clearly and legibly and show all your work, otherwise you will receive reduced credit for your answer. Read the questions carefully and make sure that you answer all the questions. You may leave your results in fractional form. Please mind your own work.

Have some fun!

1. Consider the following economy with one consumer and one firm. The consumer's utility is $u(x^1, x^2, x^3) = \sum_{i=1}^{3} \ln(x^i)$. The consumer has endowment $e = (2, 4, 8)$. The firm's technology set is $Y = \{y^3 \le 0, (y^1, y^2) \le -y^3\}$, i.e., there is joint production of goods 1 and 2 (note that it is technologically efficient for the firm to produce at the point where $y^1 = y^2 = -y^3$, but the agent/owner may choose not to produce at the technologically efficient point). Find the set of Walrasian equilibria of this economy.

2. Consider the following third-price auction with independent and private valuations. There are three risk-neutral bidders who draw their valuations from a uniform distribution on $[0, 1]$. The highest bidder wins and pays the *third* highest bid. Recall that the expected revenue from *any* IPV auction with risk neutral bidders in which the highest bidder wins is $ER = n \int_{\underline{v}}^{\overline{v}} [vf(v) - 1 + F(v)] [F(v)]^{n-1} dv$ and that the expected payment of a bidder is $P(v) = v [F(v)]^{n-1} - \int_{\underline{v}}^{v} [F(x)]^{n-1} dx$ ($F(\cdot)$ and $f(\cdot)$ are the distribution and density of valuations; in the case of the uniform distribution on $[0, 1]$, $F(x) = x$; also, you may find useful the fact that the *joint* density of two i.i.d. draws from a uniform distribution is equal to 2). Compute the expected revenue from this auction and the symmetric equilibrium bid function. To compute the equilibrium bid function, you can either compute expected equilibrium payment as the expectation of the lowest bid (conditional on winning) and set it equal to $P(v)$, or set up the problem as we did in class in the case of the first price auction; i.e, write the expected bidder surplus as a function of the bid β, and require that expected surplus be maximized at $\beta = b(v)$. How does a bidder's bid compare to his valuation? What is the optimal reserve price in this auction?

3. A monopolist with constant marginal cost $c = 1$ faces two types of consumers characterized by inverse demand functions $P_H(q_H) = 10 - 5q_H$ and $P_L(q_L) = 2 - q_L$. The monopolist knows the consumers' demands, but cannot distinguish between different types of consumers. The proportion of high type consumers is $\pi \in (0, 1)$.

a. Find the set of values of the parameter π for which the monopolist sells strictly positive quantities to both types of consumers.

b. Suppose that the monopolist sells to both types of consumers and implements an *optimal nonlinear pricing mechanism*. Find the per-unit prices and quantities that are offered by the monopolist to the low- and high-type consumers as functions of π. To do so, you must first compute the quantities that the monopolist would offer to the two types and the consumers' total outlays (payments). Dividing the total outlay by the number of objects gives the per-unit price.

4. Consider an exchange economy with two goods and 2004 agents. There are 1002 type-1 consumers with utility functions $u_i = \left(x_i^1\right)^{3/4}\left(x_i^2\right)^{1/4}$ and endowments $e_i = (1,1)$ for all $i = \overline{1, 1002}$. The remaining 1002 type-2 consumers have utility functions $u_j = \left(x_j^1\right)^{1/4}\left(x_j^2\right)^{3/4}$ and endowments $e_j = (1,1)$ for all $j = \overline{1002, 2004}$. Find a Walrasian equilibrium in this pure-exchange economy.

Suggested answers

1. Let $p^1 = 1$, $p^2 = m$, $p^3 = n$. Consumer demands are given by $x^1 = (2 + 4m + 8n)/3$, $x^2 = (2 + 4m + 8n)/(3m)$, $x^3 = (2 + 4m + 8n)/(3n)$. In addition, we have $y^1 = y^2 = -y^3$. Market clearing requires $x^1 = 2 + y^1$, $x^2 = 4 + y^2$, $x^3 = 8 + y^3$. It follows that either $m = 1/2$, $n = 1/4$ with corresponding allocation $\left(x^1, x^2, x^3, y^1, y^2, y^3\right) = (2, 4, 8, 0, 0, 0)$, or $m = \frac{\sqrt{31}-1}{6}$, $n = \frac{\sqrt{31}+5}{6}$ and corresponding allocation $\left(\frac{2(1+\sqrt{31})}{3}, \frac{2(7+\sqrt{31})}{3}, \frac{2(11-\sqrt{31})}{3}, \frac{2(1+\sqrt{31})}{3}, \frac{2(1+\sqrt{31})}{3}, -\frac{2(1+\sqrt{31})}{3}\right)$. In either case the firm's profit is equal to zero; both allocations are Walrasian equilibrium allocations, but the latter gives the consumer a higher level of utility.

2. Expected revenue is $ER = 3\int_0^1 [2v-1]\, v^2 dv = 1/2$.To compute the equilibrium bid function, note that $P(v) = v^3 - \int_0^v x^2 dx = 2v^3/3$. Consider bidder 1, and suppose his value is v_1; the payment rule specifies $p(x) = \min\{b_2, b_3\}$ if $x > \max\{b_2, b_3\}$, and zero otherwise. Therefore, bidder 1's equilibrium expected payment is $P(v_1) = \int_0^{v_1}\int_0^M b(m)\, f(m, M) dm dM$, and since M and m are independent and uniform, $f(m, M) = 2$. Thus, $P(v_1) = 2\int_0^{v_1}\int_0^M b(m)\, dm dM$. Hence, $P'(v_1) = 2\int_0^{v_1} b(m) dm = 2v_1^2$, from which $b(v_1) = 2v_1$. Alternatively, consider the expected surplus of a bidder with value v_i who bids β: $\pi(v_i, \beta) = E\left[(v_i - b(\min\{v_{-i}\}))\, |\beta \geq \max\{b(v_{-i})\}\right]$; thus, $\pi(v_i, \beta) = 2\int_0^{b^{-1}(\beta)}\int_0^M (v_i - b(m))\, dm dM$. In equilibrium we must have $\pi_2(v_i, b(v_i)) = 0$, so that, by dropping irrelevant subscripts,

$$\left.\frac{1}{b'\left(b^{-1}(\beta)\right)}\int_0^{b^{-1}(\beta)} (v - b(m))\, dm\right|_{\beta = b(v)} = 0,$$

or $\int_0^v (v - b(m))\, dm = 0$. Differentiating, we have $v - b(v) + \int_0^v dm = 0$, so that $b(v) = 2v$. Note that the bid is greater than valuation for all strictly positive valuations. The optimal reserve price in this auction is $v^* = 1/2$–the calculations above assume that there is no reserve price.

3. Normalize the number of consumers to 1. The monopolist's problem is to choose q_L, q_H to

$$\max_{q_L, q_H} \pi(X_H - q_H) + (1-\pi)(X_L - q_L)$$

s.t.

$$\begin{aligned}(IR_L) X_L &= 2q_L - \frac{q_L^2}{2} \\ (IC_H) X_H &= 10(q_H - q_L) - \frac{5}{2}\left(q_H^2 - q_L^2\right) + 2q_L - \frac{q_L^2}{2}\end{aligned}$$

(the two constraints bind at the optimum). Thus, $q_L = \frac{1-9\pi}{1-5\pi}$, $q_H = 9/5$, and for the monopolist to sell to both types we require $\pi \in (0, 1/9)$. The per-unit outlays for the low- and high-type consumers as functions of π are $p_L = X_L/q_L = \frac{3-11\pi}{2-10\pi}$ and $p_H = X_H/q_H = \frac{39+5\pi(99\pi-46)}{18(1-5\pi)^2}$.

4. The easiest way to solve the problem is to consider two (representative) agents with utility functions $u_1 = \left(x_1^1\right)^{3/4}\left(x_1^2\right)^{1/4}$ and $u_2 = \left(x_2^1\right)^{1/4}\left(x_2^2\right)^{3/4}$ and endowments $e_1 = e_2 = (1002, 1002)$. Let $p_1 = 1$ and $p_2 = p$; consumer demands are $x_1^1 = 3006\,(1+p)\,/4$, $x_1^2 = 1002\,(1+p)\,/\,(4p)$, $x_2^1 = 1002\,(1+p)\,/4$, $x_2^2 = 3006\,(1+p)\,/\,(4p)$. Market clearing requires $p = 1$, and the corresponding allocations are $\left(x_1^1, x_1^2, x_2^1, x_2^2\right) = (1503, 501, 501, 1503)$. This translates into allocations in which all type-1 agents receive $\left(x^1, x^2\right) = \left(\frac{1503}{1002}, \frac{1}{2}\right)$ and all type-2 agents receive $\left(x^1, x^2\right) = \left(\frac{1}{2}, \frac{1503}{1002}\right)$.