3D realization of two triangulations of a convex polygon.

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Abstract

We study the problem of construction of a convex 3-polytope whose (i) shadow boundary has \( n \) vertices and (ii) two hulls, upper and lower, are isomorphic to two given triangulations of a convex \( n \)-gon. Barnette [1] proved the existence of a convex 3-polytope in general case. We show that, in our case, a polytope can be constructed using an operation of edge creation.

Key words: triangulation, convex polytope, Steinitz theorem

1. Introduction

Let \( P \) be a convex polygon in the \( xy \)-plane with \( n \) vertices. Two triangulations of \( P \) are called distinct if the only edges they share are the edges of \( P \). Let \( T_1 \) and \( T_2 \) be two distinct triangulations of \( P \). At the First Canadian Conference on Computational Geometry Leo Guibas conjectured that it is always possible to perturb the vertices of \( P \) vertically out (i.e., by displacements parallel to the \( z \)-axis) so that the polygon \( P \) becomes a spatial polygon \( P' \) such that the convex hull of \( P' \) is a convex polyhedron consisting of two triangulated cups glued along \( P' \), and the triangulation of the upper cup (i.e., those faces oriented toward \( +z \)) is that specified as \( T_1 \) and the triangulation of the lower cup is that specified as \( T_2 \) [4].

Boris Bekster [2] disproved Guibas’ conjecture by showing a counterexample, a convex hexagon with two triangulations. Marlin and Toussaint [3] considered the computational problem of deciding whether a triple \( (P, T_1, T_2) \) admits a realization in \( \mathbb{R}^3 \). They reduced the problem to a linear programming problem with \( O(n^2) \) inequality constraints and \( n \) variables. The variables are \( z \)-coordinates of lifted vertices of \( P \) and the constraints correspond to vertex-face relations; the vertices must be below/above the planes passing through faces of the upper/lower cup of \( P' \). The number of constraints can be dropped to \( 2n - 6 = |T_1| + |T_2| \) by considering dihedral angles corresponding to diagonals of the triangulations [7].

Guibas conjecture is related to Steinitz’s theorem [5].

Steinitz’s Theorem: A graph \( G \) is isomorphic to the edge graph of a convex 3-polytope if and only if \( G \) is 3-connected and planar.

By Steinitz’s theorem the graph \( (P, T_1 \cup T_2) \) is the edge graph of a convex 3-polytope [3]. According to Barnette’s theorem [1], every 3-polytope with a Hamiltonian circuit has realization such that the Hamiltonian circuit is a shadow boundary. This implies that Guibas’ conjecture is true up to a combinatorial deformation [2]. Formally this can be stated as follows.

Theorem 1 For any two distinct triangulations \( T_1 \) and \( T_2 \) of a convex polygon \( P_2 \) in \( \mathbb{R}^2 \) with \( n \) vertices, there is a convex polytope \( P_3 \) in \( \mathbb{R}^3 \) with \( n \) vertices such that (i) the \( xy \)-shadow \( S \) of \( P_3 \) contains all its vertices, and (ii) there is an isomorphism \( \tau : P_2 \to S \) that maps the edges of \( T_1 \) (resp. \( T_2 \)) to the edges of the upper hull of \( P_3 \) (resp. the lower hull).

Barnette’s proof deals with general faces (not just triangles) due to its generality. In this paper we give a different proof of Theorem 1 that uses only triangular faces of polytopes which can be turned into a more robust algorithm for finding a combinatorial realization of \( (P, T_1, T_2) \) in \( \mathbb{R}^3 \).

Realization questions have been studied in com-
puter graphics and scene analysis as well. Sugihara [6] established necessary and sufficient conditions whether a line drawing in the plane can be realized in \( \mathbb{R}^3 \) by lifting.

We call a triple \( (P, T_1, T_2) \) a configuration. We call a map \( \tau \) satisfying the conditions of Theorem 1 a realization.

2. Edge contraction

Fig. 1 (a) for example where the edge \( p_1p_2 \) is contracted. When applied for two triangulations, we want the reduced triangulations to be distinct. An edge \( e \) of a configuration \( (P, T_1, T_2) \) is contractible if the new triangulations \( T_1' \) and \( T_2' \) are distinct. In general, not all edges are contractible. For example, the edge \( p_1p_2 \) in the Fig. 2 (a) is not contractible since two edges \( p_1p_6 \) and \( p_2p_6 \) from different triangulations coincide after the contraction of \( p_1p_2 \).

As in Barnette’s proof we use the operation of edge removal. The difference is that we will not apply it for diagonals of \( P \). This prevents the appearance of faces with more than three vertices. The edge contraction in a configuration is defined by identifying the edge endpoints. If applied to one triangulation of \( P \), it produces a triangulation, see

![Diagram](image1)

(a) Edge contraction of a triangulation. (b) Edge contraction of two triangulations. The diagonals of one triangulation are solid and the diagonals of the other triangulation are dashed.

![Diagram](image2)

(a) Fig. 1. (a) Edge contraction for \( n = 4 \).

**Lemma 2** Let \( C \) be a configuration with \( n \geq 4 \) vertices. There is a contractible edge of \( C \) among the edges of the convex polygon.

**Proof.** If \( n = 4 \) then every edge of the convex polygon is contractible, see Fig. 2 (b). We prove the lemma for \( n \geq 5 \). Suppose to the contrary that there is a configuration \( (P, T_1, T_2) \) such that all edges of \( P \) are not contractible. Let \( p_1, \ldots, p_n \) be
the vertices of \( P \) in clockwise order. The edge \( p_1p_2 \)
is not contractible. Then there is a vertex \( p_k, 1 \leq k \leq n - 1 \)such that \( p_1, p_k \) is an edge of one triangulation, say \( T_1 \), and \( p_2p_k \) is an edge of \( T_2 \), see Fig. 3 (a).

Consider an edge \( p_ip_{i+1}, 2 \leq i \leq k - 1 \). Since
\( p_ip_{i+1} \) is not contractible, there is a vertex \( p_c(i) \)such that \( p_ip_c(i) \) is a diagonal of \( T_j, j = 1, 2 \) and
\( p_{i+1}p_c(i) \) is a diagonal of \( T_{3-j} \), see Fig. 3 (a). We
call \( p_c(i) \) a witness since it indicates that \( p_ip_{i+1} \) is
not contractible. At least one vertex of \( \{p_i, p_{i+1}\} \),say \( p_i \), is different from \( p_j \) and \( p_k \). Then the edge
\( p{j}p_c(i) \) does not cross one of the edges \( p_1p_k \) or \( p_2p_k \).
Therefore \( c(i) \) is an index in the range \( 1, \ldots, k \).

\[ \text{Fig. 3. Lemma 2.} \]

We call \( p_c(i) \) a left witness if \( c(i) < i \). We call \( p_c(i) \)
a right witness if \( c(i) > i+1 \). Each witness is either
left or right since \( c(i) \neq i, i+1 \). Note that \( p_c(2) \) is
a right witness and \( p_c(k-1) \) is a left witness. Thus
there is an index \( i, 2 \leq i \leq k - 1 \) such that \( p_c(i) \) is
the right index and \( p_c(i+1) \) is the left index, see Fig. 3 (b).
Then \( p_ip_c(i), a diagonal of a triangulation \( T_j \),
intersects both diagonals \( e_1 = (p_{i+1}, p_c(i+1)) \) and
\( e_2 = (p_{i+2}, p_c(i+1)) \). Either \( e_1 \) or \( e_2 \) is a diagonal of
\( T_j \). Contradiction.

3. Edge creation

We define an operation of edge creation as the reverse operation of the edge contraction. The fol-
lowing lemma characterizes the change of the configuration when an edge is created. We denote the
sequence of indices from \( i \) to \( j \) in clockwise order by \( \{i, i+1, \ldots, j\} \).

**Lemma 3 (Edge creation)** Let \( \mathcal{C} = (P, T_1, T_2) \)
be a configuration with \( n \geq 3 \) vertices where \( P = \{p_1, \ldots, p_n\} \). Suppose that an edge \( e = (q_1, q_2) \)
is created in place of a vertex \( p_i \) \( \in \ P \). Let \( \mathcal{C'} = (P', T_1', T_2') \) be the configuration obtained by replacing
a vertex \( p_i \) by an edge \( e = (q_1, q_2) \) in clockwise
order. Then there are two edges \( (p_i, p_j) \in T_1 \) and
\( (p_i, p_k) \in T_2 \) such that
- an edge \( (p_i, p_j) \in T_1, j \in \{i+1, i+2, \ldots, j\} \) is
  replaced by the edge \( (p_i, q_1) \in T_1' \) and
- an edge \( (p_i, p_k) \in T_2, k \in \{j, j+1, \ldots, i-1\} \) is
  replaced by the edge \( (p_i, q_2) \in T_2' \) and
- an edge \( (p_k, p_j) \in T_2, j \in \{i+1, i+2, \ldots, k\} \) is
  replaced by the edge \( (q_1, q_2) \in T_2' \) and
- an edge \( (p_k, p_i) \in T_2, j \in \{k, k+1, \ldots, i-1\} \) is
  replaced by the edge \( (p_i, q_1) \in T_1' \) and

We show that an edge can be always created.

**Theorem 4** Let \( \mathcal{C} = (P, T_1, T_2) \) be a configuration
with \( n \geq 3 \) vertices and let \( \mu : P \to \mathbb{R}^3 \) be its re-
alization in \( \mathbb{R}^3 \). Let \( \mathcal{C'} = (P', T_1', T_2') \) be the con-
figuration obtained by an edge creation. There is a
realization of \( \mathcal{C'} \) if there is a realization of \( \mathcal{C} \).

Theorem 1 follows from Theorem 4.

**References**


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Fig. 4. Edge creation.


