• The Priority Queue Abstract Data Type

• Implementing A Priority Queue With a Sequence
Keys and Total Order Relations

- A Priority Queue ranks its elements by key with a total order relation

• Keys:
  - Every element has its own key
  - Keys are not necessarily unique

• Total Order Relation
  - Denoted by \( \leq \)
  - Reflexive: \( k \leq k \)
  - Antisymmetric: if \( k_1 \leq k_2 \) and \( k_2 \leq k_1 \), then \( k_1 \leq k_2 \)
  - Transitive: if \( k_1 \leq k_2 \) and \( k_2 \leq k_3 \), then \( k_1 \leq k_3 \)

• A Priority Queue supports these fundamental methods:
  - `insertItem(k, e)` // element e, key k
  - `removeMinElement()` // return and remove the // item with the smallest key
A Priority Queue $P$ can be used for sorting by inserting a set $S$ of $n$ elements and calling $\text{removeMinElement()}$ until $P$ is empty:

**Algorithm** PriorityQueueSort($S$, $P$):

*Input*: A sequence $S$ storing $n$ elements, on which a total order relation is defined, and a Priority Queue $P$ that compares keys with the same relation

*Output*: The Sequence $S$ sorted by the total order relation

```
while !S.isEmpty() do
    e ← S.removeFirst()
    P.insertItem(e, e)
while P is not empty do
    e ← P.removeMinElement()
    S.insertLast(e)
```
The Priority Queue ADT

- A priority queue \( P \) must support the following methods:

- **size()**: Return the number of elements in \( P \)
  
  **Input**: None;  
  **Output**: integer

- **isEmpty()**: Test whether \( P \) is empty
  
  **Input**: None;  
  **Output**: boolean

- **insertItem\((k,e)\)**: Insert a new element \( e \) with key \( k \) into \( P \)
  
  **Input**: Objects \( k, e \);  
  **Output**: None

- **minElement()**: Return (but don’t remove) an element of \( P \) with smallest key; an error occurs if \( P \) is empty.
  
  **Input**: None;  
  **Output**: Object \( e \)
The Priority Queue ADT (contd.)

- **minKey():**
  Return the smallest key in $P$; an error occurs if $P$ is empty
  **Input:** None; **Output:** Object $k$

- **removeMinElement():**
  Remove from $P$ and return an element with the smallest key; an error condition occurs if $P$ is empty.
  **Input:** None; **Output:** Object $e$
Comparators

• The most general and reusable form of a priority queue makes use of comparator objects.

• Comparator objects are external to the keys that are to be compared and compare two objects.

• When the priority queue needs to compare two keys, it uses the comparator it was given to do the comparison.

• Thus a priority queue can be general enough to store any object.

• The comparator ADT includes:
  - isLessThan\((a, b)\)
  - isLessThanOrEqualTo\((a, b)\)
  - isEqualTo\((a, b)\)
  - isGreaterThan\((a, b)\)
  - isGreaterThanOrEqualTo\((a, b)\)
  - isComparable\((a)\)
Implementation with an Unsorted Sequence

• Let’s try to implement a priority queue with an unsorted sequence $S$.

• The elements of $S$ are a composition of two elements, $k$, the key, and $e$, the element.

• We can implement `insertItem()` by using `insertFirst()` of the sequence. This would take $O(1)$ time.

• However, because we always `insertFirst()`, despite the key value, our sequence is not ordered.
Implementation with an Unsorted Sequence (contd.)

• Thus, for methods such as `minElement()`, `minKey()`, and `removeMinElement()` , we need to look at all elements of $S$. The worst case time complexity for these methods is $O(n)$. 
Implementation with a Sorted Sequence

- Another implementation uses a sequence $S$, sorted by keys, such that the first element of $S$ has the smallest key.

- We can implement `minElement()`, `minKey()`, and `removeMinElement()` by accessing the first element of $S$. Thus these methods are $O(1)$ (assuming our sequence has an $O(1)$ front-removal)

- However, these advantages comes at a price. To implement `insertItem()`, we must now scan through the entire sequence. Thus `insertItem()` is $O(n)$.
public class SequenceSimplePriorityQueue implements SimplePriorityQueue {

    //Implementation of a priority queue using a sorted sequence
    protected Sequence seq = new NodeSequence();
    protected Comparator comp;

    // auxiliary methods
    protected Object extractKey (Position pos) {
        return ((Item)pos.element()).key();
    }

    protected Object extractElem (Position pos) {
        return ((Item)pos.element()).element();
    }

    protected Object extractElem (Object key) {
        return ((Item)key).element();
    }

    // methods of the SimplePriorityQueue ADT
    public SequenceSimplePriorityQueue (Comparator c) {
        this.comp = c;
    }

    public int size () {return seq.size();}
}
public boolean isEmpty () { return seq.isEmpty(); }
public void insertItem (Object k, Object e) throws InvalidKeyException {
    if (!comp.isComparable(k))
        throw new InvalidKeyException("The key is not valid");
    else
        if (seq.isEmpty())
            seq.insertFirst(new Item(k,e));
        else
            if (comp.isGreaterThan(k,extractKey(seq.last())))
                seq.insertAfter(seq.last(),new Item(k,e));
            else {
                Position curr = seq.first();
                while (comp.isGreaterThan(k,extractKey(curr))
                    curr = seq.after(curr);
                seq.insertBefore(curr,new Item(k,e));
            }
}
public Object minElement () throws EmptyContainerException {
    if (seq.isEmpty())
        throw new EmptyContainerException("The priority queue is empty");
    else
        return extractElem(seq.first());
}
Selection Sort

- Selection Sort is a variation of PriorityQueueSort that uses an unsorted sequence to implement the priority queue P.

- **Phase 1**, the insertion of an item into P takes $O(1)$ time.

- **Phase 2**, removing an item from P takes time proportional to the number of elements in P

<table>
<thead>
<tr>
<th>Phase 1:</th>
<th>Sequence S</th>
<th>Priority Queue P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td>(7, 4, 8, 2, 5, 3, 9)</td>
<td>()</td>
</tr>
<tr>
<td>Phase 1:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a)</td>
<td>(4, 8, 2, 5, 3, 9)</td>
<td>(7)</td>
</tr>
<tr>
<td>(b)</td>
<td>(8, 2, 5, 3, 9)</td>
<td>(7, 4)</td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(g)</td>
<td>()</td>
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<table>
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<tr>
<th>Phase 2:</th>
<th>Sequence S</th>
<th>Priority Queue P</th>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a)</td>
<td>(2)</td>
<td>(7, 4, 8, 5, 3, 9)</td>
</tr>
<tr>
<td>(b)</td>
<td>(2, 3)</td>
<td>(7, 4, 8, 5, 9)</td>
</tr>
<tr>
<td>(c)</td>
<td>(2, 3, 4)</td>
<td>(7, 8, 5, 9)</td>
</tr>
<tr>
<td>(d)</td>
<td>(2, 3, 4, 5)</td>
<td>(7, 8, 9)</td>
</tr>
<tr>
<td>(e)</td>
<td>(2, 3, 4, 5, 7)</td>
<td>(8, 9)</td>
</tr>
<tr>
<td>(f)</td>
<td>(2, 3, 4, 5, 7, 8)</td>
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Selection Sort (cont.)

- As you can tell, a bottleneck occurs in Phase 2. The first `removeMinElement` operation takes \( O(n) \), the second \( O(n-1) \), etc. until the last removal takes only \( O(1) \) time.

- The total time needed for phase 2 is:
  \[
  O(n + (n - 1) + \ldots + 2 + 1) = O \left( \sum_{i=1}^{n} i \right)
  \]

- By a common proposition:
  \[
  \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
  \]

- The total time complexity of phase 2 is then \( O(n^2) \). Thus, the time complexity of the algorithm is \( O(n^2) \).
Insertion Sort

- Insertion sort is the sort that results when we perform a PriorityQueueSort implementing the priority queue with a sorted sequence.

- We improve phase 2 to $O(n)$.

- However, phase 1 now becomes the bottleneck for the running time. The first `insertItem` takes $O(1)$, the second $O(2)$, until the last operation takes $O(n)$.

- The run time of phase 1 is $O(n^2)$ thus the run time of the algorithm is $O(n^2)$. 
• Selection and insertion sort both take $O(n^2)$.

• Selection sort will always take $\Omega(n^2)$ time, no matter the input sequence.

• The run of insertion sort varies depends on the input sequence.

• We have yet to see the ultimate priority queue....

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Heaps

• A **Heap** is a Binary Tree \( H \) that stores a collection of keys at its internal nodes and that satisfies two additional properties:
  - 1) **Heap-Order Property**
  - 2) **Complete Binary Tree Property**

• **Heap-Order Property Property (Relational):** In a heap \( H \), for every node \( v \) (except the root), the key stored in \( v \) is greater than or equal to the key stored in \( v \)'s parent.

• **Complete Binary Tree Property (Structural):** A Binary Tree \( T \) is complete if each level but the last is full, and, in the last level, all of the internal nodes are to the left of the external nodes.
Heaps (contd.)

• An Example:
Height of a Heap

- **Proposition:** A heap $H$ storing $n$ keys has height
  \[ h = \lceil \log(n+1) \rceil \]

- **Justification:** Due to $H$ being complete, we know:
  - # $i$ of internal nodes is at least:
    \[ 1 + 2 + 4 + \ldots + 2^{h-2} + 1 = 2^{h-1} - 1 + 1 = 2^{h-1} \]
  - # $i$ of internal nodes is at most:
    \[ 1 + 2 + 4 + \ldots + 2^{h-1} = 2^{h} - 1 \]
  - Therefore:
    \[ 2^{h-1} \leq n \text{ and } n \leq 2^{h} - 1 \]
  - Which implies that:
    \[ \log(n + 1) \leq h \leq \log n + 1 \]
  - Which in turn implies:
    \[ h = \lfloor \log(n+1) \rfloor \]
  - Q.E.D.
Heigh of a Heap (contd.)

- Let’s look at that graphically:

Consider this heap which has height $h = 4$ and $n = 13$

Suppose two more nodes are added. To maintain completeness of the tree, the two external nodes in level 4 will become internal nodes: i.e. $n = 15, h = 4 = \log(15+1)$

Add one more: $n = 16, h = 5 = \lceil \log(16+1) \rceil$
Insertion into a Heap (cont.)

Priority Queues
Insertion into a Heap (cont.)

(4,C)
(5,A)
(15,K)
(16,X)
(15,K)
(25,J)
(14,E)
(12,H)
(11,S)
(8,W)
(7,Q)
(6,Z)
(2,T)
(20,B)
(4,C)
(5,A)
(15,K)
(16,X)
(25,J)
(14,E)
(12,H)
(11,S)
(8,W)
(7,Q)
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(2,T)
(20,B)
Insertion into a Heap (cont.)

Priority Queues
Removal from a Heap

(4,C)  (13,W)

(5,A)  (6,Z)

(15,K)  (9,F)

(25,J) (14,E) (12,H)

(16,X) (11,S)
Removal from a Heap (cont.)

(13, W)
Removal from a Heap (cont.)
Removal from a Heap (cont.)

Priority Queues
Implementation of a Heap

public class HeapSimplePriorityQueue implements SimplePriorityQueue {
    BinaryTree T;
    Position last;
    Comparator comparator;
    ...
}

Priority Queues
Implementation of a Heap (cont.)

- Two ways to find the insertion position $z$ in a heap:

a)

b)
Heap Sort

- All heap methods run in logarithmic time or better
- If we implement PriorityQueueSort using a heap for our priority queue, `insertItem` and `removeMinElement` each take $O(\log k)$, $k$ being the number of elements in the heap at a given time.
- We always have $n$ or less elements in the heap, so the worst case time complexity of these methods is $O(\log n)$.
- Thus each phase takes $O(n\log n)$ time, so the algorithm runs in $O(n\log n)$ time also.
- This sort is known as heap-sort.
- The $O(n\log n)$ run time of heap-sort is much better than the $O(n^2)$ run time of selection and insertion sort.
Bottom-Up Heap Construction

• If all the keys to be stored are given in advance we can build a heap bottom-up in $O(n)$ time.

• Note: for simplicity, we describe bottom-up heap construction for the case for $n$ keys where:

$$n = 2^h - 1$$

$h$ being the height.

• Steps:
  1) Construct $(n+1)/2$ elementary heaps with one key each.
  2) Construct $(n+1)/4$ heaps, each with 3 keys, by joining pairs of elementary heaps and adding a new key as the root. The new key may be swapped with a child in order to preserve heap-order property.
  3) Construct $(n+1)/8$ heaps, each with 7 keys, by joining pairs of 3-key heaps and adding a new key. Again swaps may occur.
  ...
  4) In the $i$th step, $2 \leq i \leq h$, we form $(n+1)/2^i$ heaps, each storing $2^i - 1$ keys, by joining pairs of heaps storing $(2^{i-1} - 1)$ keys. Swaps may occur.
Bottom-Up Heap Construction (cont.)
Bottom-Up Heap Construction (cont.)
Bottom-Up Heap Construction (cont.)
Bottom-Up Heap Construction (cont.)

The End
Analysis of Bottom-Up Heap Construction

- **Proposition**: Bottom-up heap construction with $n$ keys takes $O(n)$ time.
  - Insert $(n + 1)/2$ nodes
  - Insert $(n + 1)/4$ nodes
  - Upheap at most $(n + 1)/4$ nodes 1 level.
  - Insert $(n + 1)/8$ nodes
  - ...
  - Insert 1 node.
  - Upheap at most 1 node 1 level.

- $n$ inserts, $n/2$ upheaps of 1 level = $O(n)$
Locators

- Locators can be used to keep track of elements in a container
- A locator sticks with a specific key-element pair, even if that element “moves around”.
- The Locator ADT supports the following fundamental methods:

  - **element()**: Return the element of the item associated with the Locator.
    - **Input**: None; **Output**: Object
  
  - **key()**: Return the key of the item associated with the Locator.
    - **Input**: None; **Output**: Object
  
  - **isContained()**: Return true if and only if the Locator is associated with a container.
    - **Input**: None; **Output**: boolean
  
  - **container()**: Return the container associated with the Locator.
    - **Input**: None; **Output**: boolean
Priority Queue with Locators

• It is easy to extend the sequence-based and heap-based implementations of a Priority Queue to support Locators.

• The Priority Queue ADT can be extended to implement the Locator ADT

• In the heap implementation of a priority queue, we store in the locator object a key-element pair and a reference to its position in the heap.

• All of the methods of the Locator ADT can then be implemented in $O(1)$ time.
A Java Implementation of a Locator

```java
public class LocItem extends Item implements Locator {
    private Container cont;
    private Position pos;

    LocItem (Object k, Object e, Position p, Container c) {
        super(k, e);
        pos = p;
        cont = c;
    }

    public boolean isContained() throws InvalidLocatorException {
        return cont != null;
    }

    public Container container() throws InvalidLocatorException {
        return cont;
    }

    protected Position position() {
        return pos;
    }

    protected void setPosition(Position p) {
        pos = p;
    }

    protected void setContainer(Container c) {
        cont = c;
    }
}
```
A Java Implementation of a Locator-Based Priority Queue

```java
public class SequenceLocPriorityQueue extends SequenceSimplePriorityQueue implements PriorityQueue {
    // priority queue with locators implemented with a sorted sequence
    public SequenceLocPriorityQueue (Comparator comp) {
        super(comp);
    }

    // auxiliary methods
    protected LocItem locRemove(Locator loc) {
        checkLocator(loc);
        seq.remove(((LocItem) loc).position());
        ((LocItem) loc).setContainer(null);
        return (LocItem) loc;
    }
}
```
protected Locator locInsert(LocItem locit) throws InvalidKeyException {
    Position p, curr;
    Object k = locit.key();
    if (!comp.isComparable(k))
        throw new InvalidKeyException("The key is not valid");
    else if (seq.isEmpty())
        p = seq.insertFirst(locit);
    else if (comp.isGreaterThan(k, extractKey(seq.last())))
        p = seq.insertAfter(seq.last(), locit);
    else {
        curr = seq.first();
        while (comp.isGreaterThan(k, extractKey(curr)))
            curr = seq.after(curr);
        p = seq.insertBefore(curr, locit);
    }
    locit.setPosition(p);
    locit.setContainer(this);
    return (Locator) locit;
}
Priority Queues

Locator-Based PQ (contd.)

```java
public void insert(Locator loc) throws InvalidKeyException {
    locInsert((LocItem) loc);
}

public Locator insert(Object k, Object e) throws InvalidKeyException {
    LocItem locit = new LocItem(k, e, null, null);
    return locInsert(locit);
}

public void insertItem (Object k, Object e) throws InvalidKeyException {
    insert(k, e);
}

public void remove(Locator loc) throws InvalidLocatorException {
    locRemove(loc);
}

public Object removeMinElement () throws EmptyContainerException {
    Object toReturn = minElement();
    remove(min());
    return toReturn;
}
```