Dynamic Optimization of an Oligopoly Model of Advertising

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Abstract.
We examine an oligopoly model of advertising competition where each firm’s market share depends on its own and its competitors’ advertising decisions. A differential game model is developed and used to derive the closed-loop Nash equilibrium under symmetric as well as asymmetric competition. We obtain explicit solutions under certain plausible conditions, and discuss the effects of an increase in the number of competing firms on advertising expenditure, market share and profitability.

Key Words: Advertising, Oligopoly, Differential games, Optimization.
1. Introduction

Many industries are characterized by firms competing for market share primarily on the basis of advertising. The markets for cola drinks, beer and cigarettes are some examples that have been studied in the literature (Erickson 1992, Fruchter and Kalish 1997). Each firm’s advertising acts to increase its market share while the competitors’ advertising acts to reduce its market share. Owing to the carry-over dynamics of advertising, an analytical solution in continuous time typically starts by modeling the competitive interaction as a differential game, where each firm tries to increase its profit while taking into consideration the response of its competitors. Whereas elements of the marketing environment such as carry-over dynamics, competition, competitive and non-competitive decay have been described by individual models, understanding optimal advertising policies in differential games is hindered by the difficulty in obtaining explicit solutions.

Despite this difficulty, a few intrepid researchers have examined the important issue of dynamic advertising competition involving more than two firms. Teng and Thompson (1983) and Dockner and Jorgensen (1992) develop oligopoly models where sales rather than market share is the dependent variable and competitive advertising affects sales indirectly by its effect on market saturation. These models are based on the innovation diffusion dynamics in which the market reaches saturation after some time. They show that advertising should decrease over time, caused possibly by the saturation effect. Using an advertising model based on goodwill accumulation, Fershtman (1984) studied oligopoly competition and obtained the following results: (i) Firms with lower production costs obtain larger market share. (ii) As the number of firms increases, individual firms’ equilibrium advertising decreases, except possibly for the market share leader. (iii) If all firms are identical, the firms’ equilibrium advertising tends to
decrease with interest rate, depreciation rate, and production cost increases. We will examine whether these results continue to hold in our setting, and find that whereas some hold, others do not.

Erickson (1995, 2003) uses the method of dynamic conjectural variations to study oligopoly markets. For the case where the discount rate is zero, and for three symmetric competitors, Erickson (2003, p.103) obtains a simple solution for the advertising rate, showing that it increases with the profit margin and the conjectured lack of response of the competitor, and it decreases with the concavity of the market shares response to advertising. We will examine the effects of margin and competitive response in our setting. Fruchter (1999) extends the closed-loop duopoly analysis of Fruchter and Kalish (1987) to an oligopoly, and demonstrates that treating an oligopoly as a two-player game by aggregating all the rival firms results in suboptimal advertising. Particularly for the papers that use closed-loop Nash equilibrium solutions, simulations are needed to understand the effect of parameter changes, whereas in our model, at least for symmetric firms, explicit solutions are obtained that can be analytically handled.\(^2\)

In this paper we propose to extend the work of Sethi (1983) to an oligopoly. In the original paper, Sethi (1983) proposed a stochastic model of advertising dynamics for a monopoly that admitted explicit solutions. As we shall see, the extension is appropriate only when there is not too great an asymmetry between firms. Nevertheless, it is valuable in providing, for the first time, explicit closed-loop solutions for the oligopoly case. We will solve an infinite horizon,

\(^2\) Differential games can be solved using either open-loop or closed-loop solution concepts. In the open-loop solution, competing firms decide at inception what their advertising expenditures will be over the planning horizon. The closed-loop solution envisages that competing firms decide upon their advertising response given the current state. Whereas the latter concept, being time consistent, is intuitively more appealing, is robust, provides a better fit to empirical data and satisfies subgame perfection, it is more difficult to compute than an open-loop solution (e.g., Chintagunta 1993, Erickson 1995, 1992, Feichtinger et. al 1994, Fruchter and Kalish 1997). Typically, resort must be made to numerical methods of solution.
differential game model using the closed-loop Nash equilibrium concept to obtain insights into the effects of an increase in the number of competing firms on advertising expenditure, market share and profitability.

The rest of the paper is divided into sections dealing with the Sethi model that provides the conceptual underpinnings for the present effort, the description of the proposed model, the analysis for symmetric and asymmetric firms, and finally, the conclusions.

2. The Sethi Model and its Extensions

The original Sethi model is stochastic but, for our purposes, the deterministic version given by

\[ \frac{dx(t)}{dt} = \rho u(t)\sqrt{1-x(t)} - \delta x(t), \quad x(0) = x_0, \quad (1) \]

is relevant, where \( x(t) \) is the sales rate (expressed as a fraction of the total market) at time \( t \), \( u(t) \) is the advertising expenditure rate, \( \rho \) is a response constant and \( \delta \) is a market share decay constant. The parameter \( \rho \) may be conceptualized as brand strength, which determines the effectiveness of advertising, while \( \delta \) determines the rate at which consumers are lost due to product obsolescence, forgetting, etc. The formulation, like the classical Vidale-Wolfe (1957) model, has the desirable properties that market share has a concave response to advertising and there is a saturation level (Little 1979).

Sorger (1989) combined the Sethi model with the Lanchester framework to obtain

\[ \frac{dx}{dt} = \rho_1 u_1(x, y)\sqrt{1-x} - \rho_2 u_2(x, y)\sqrt{x}, \quad x(0) = x_0, \quad (2) \]

\[ \frac{dy}{dt} = \rho_2 u_2(x, y)\sqrt{1-y} - \rho_1 u_1(x, y)\sqrt{y}, \quad y(0) = 1-x_0, \quad (3) \]
where \( x(t) \) and \( y(t) \) represent the market shares of the two firms, whose parameters and decision variables are indexed 1 and 2, respectively. Note that \( x(t) + y(t) = 1 \). Sorger describes some appealing characteristics of the model in detail, noting that it is compatible with word-of-mouth and nonlinear effects, and provides a comparison with other dynamics used in the advertising scheduling literature. Moreover, Chintagunta and Jain (1995) have provided empirical support for the Sethi model by testing Sorger’s specification using data from the pharmaceuticals, soft drinks, beer and detergent industries, and find it to be appropriate.

Prasad and Sethi (2003) consider a duopoly extension of the Sethi model including the stochastic noise. The deterministic part of their dynamics differs from Sorger’s model in having a churn parameter \( \delta \), and is given by

\[
\begin{align*}
\frac{dx}{dt} &= \rho_1 u_1(x, y)\sqrt{1-x} - \rho_1 u_2(x, y)\sqrt{x} - \delta(x-y), \quad x(0) = x_0, \quad (4) \\
\frac{dy}{dt} &= \rho_2 u_2(x, y)\sqrt{1-y} - \rho_2 u_1(x, y)\sqrt{y} - \delta(y-x), \quad y(0) = 1-x_0. \quad (5)
\end{align*}
\]

They note that churn is caused by non-competitive factors such as product obsolescence, forgetting, lack of market differentiation, lack of information, variety seeking and brand switching behavior. It acts to equalize market shares of the competing firms.

In all these cases, the objective for firm \( i \) is given by

\[
\begin{align*}
\text{Max}_{a_{i,20}} \left\{ V_i(x_0) = \int_0^\infty e^{-\gamma t} [m_i x_i(t) - c_i u_i(t)] dt \right\}, \quad (6)
\end{align*}
\]

where \( c_i \) is a cost parameter and \( V_i \) is the profit or value function.\(^3\)

\(^3\) When advertising expenditure enters linearly in the dynamic equation, its cost in the objective function is often assumed to be quadratic (e.g., Erickson 1995). Equivalently, one can take the square root of the advertising expenditure in the dynamic equation and subtract advertising expenditure linearly in the objective function (e.g., Sorger 1989). See Sethi and Thompson (2000) for a discussion. However, if empirical evidence suggests convexity of the objective function, see the literature on chattering or pulsing advertising policies.
These models, and several others such as Deal (1979), Deal, Sethi and Thompson (1979), Erickson (1992), Horsky and Mate (1988), provide excellent insight into advertising spending under monopoly and duopolistic competition. This literature is surveyed by Ericsson (2003), Feichtinger, Hartl and Sethi (1994), Jorgensen (1982) Jorgensen and Zaccour (2004), and Sethi (1977). However, researchers have been concerned about the paucity of results for the general n-firm oligopoly. Indeed, with the notable exception of Ericsson (2003), there have been few attempts to study even a triopoly due to the intractability of the analysis. Nevertheless, it cannot be ignored that many industries have more than two competitors.

3. Model

We consider an \( n \)-firm oligopoly market in a mature product category so that the total sales of the category are relatively stable. Let \( x_i(t) \) denote the market share of firm \( i \), \( i \in \{1, 2, \ldots, n\} \) at time \( t \) and \( n \geq 2 \). We will use notation listed in Table 1.

\[
\begin{align*}
\text{Firm } i \text{'s objective is to maximize its long-run, discounted profits: } & \\
\max_{x_0} \left\{ 
\int_0^\infty e^{-\delta t} [m_i x_i(t) - c_i x_i(t)^2] dt \right\}. & \quad (7) \\
\end{align*}
\]

This is subject to the dynamics given by

\[
\frac{dx_i}{dt} = \underbrace{\rho_i u_i \sqrt{1-x_i}}_{\text{Gain from competitors}} - \sum_{j=1, j \neq i}^{n} \underbrace{\xi_j u_j \sqrt{1-x_j}}_{\text{Loss to competitors}} - \underbrace{\delta(x_i - \frac{1}{n} \sum_{k=1}^{n} x_k)}_{\text{Market share churn}}, \quad \forall i \in I. \quad (8)
\]

The proposed dynamics is the oligopoly version of the Sethi model where advertising influences non-adopters of the product to purchase the advertised brand, and churn acts to
equalize competitors’ market shares. When the churn term is omitted, the dynamics is also identical to Sorger (1989) for the duopoly case, which is desirable since the Sorger model has been empirically validated by Chintagunta and Jain (1995). As noted by Sorger (1989), the dynamics has additional desirable properties such as resembling the basic Vidale-Wolfe and Lanchester models and having word-of-mouth effects. The latter is due to the expansion \( \sqrt{1-x_i} = 1 - x_i + x_i(1-x_i) \), where the last term is interpreted as an interaction between consumers. In equilibrium, the dynamics resembles an excess advertising model.

It is required that market shares should not become negative. We will obtain a sufficient condition to ensure that this constraint is met given equilibrium advertising strategies. The analysis thus applies to oligopoly markets where this condition is met. There is also a logical consistency requirement that the market shares should sum to one, i.e.,

\[
\sum_{i=1}^{n} x_i = 1, \quad (9)
\]

implying that \( \sum_{i=1}^{n} \frac{dx_i}{dt} = 0 \). This imposes the restriction on the parameters

\[
\xi_i = \rho_i / (n-1), \quad \forall i \in I. \quad (10)
\]

Equation (8) can be written after inserting (10) as

\[
\frac{dx_i}{dt} = \frac{n}{n-1} \rho_i u_i \sqrt{1-x_i} - \frac{1}{n-1} \sum_{j \neq i} \rho_j u_j \sqrt{1-x_j} - \delta(x_i - \frac{1}{n}), \quad \forall i \in I. \quad (11)
\]

4. Analysis

The analysis is performed by obtaining and solving the Hamilton-Jacobi-Bellman (HJB) equation for each firm. We first state two assumptions, where the conditions for the assumptions to hold will be determined in later sections.
Assumption A1: The optimal advertising control $u_i$ is nonzero and positive for all firms.

An examination of the objective function shows that it is quadratic in the control variable and hence negative advertising expenditures are automatically excluded. This is because the cost term $c_iu_i^2$ is still positive when $u_i$ is negative. However, there may be cases where advertising is zero so we first examine the case where it is nonzero.

Assumption A2: The market share of each firm is bounded between 0 and 1.

It will be shown that the nature of the dynamics ensures that market share never exceeds 1. We will derive the situations for market shares to remain nonnegative given optimal advertising decisions in a later section. This approach is preferable to constraining the market shares ex ante since the constrained maximization problem is intractable to solve.

In Theorem 1 below, we provide the Nash Equilibrium solution of the advertising game. (Proofs are in the Appendix).

**Theorem 1:** For the differential game given by equations (7) and (11), when (A1) and (A2) hold, the optimal feedback advertising for firm $i$ is given by

$$u_i^* = \frac{P_i\sqrt{1-x_i}}{2c_i(n-1)}\left(n\phi_i - \sum_{k=1}^{n} \phi_k\right), \quad \forall i \in \mathbf{I},$$

(12)

and the value function is

$$V_i = \phi_i^0 + \sum_{j=1}^{n} \phi_j x_j, \quad \forall i \in \mathbf{I},$$

(13)

where the unknown parameters are determined from the relations
The noteworthy aspect here is that the HJB equations could be solved to yield a relatively simple optimal control, which is a consequence of the linear value function solution.

4.1. Illustration

To illustrate the application of Theorem 1, we examine the case of a triopoly. To simplify the exposition, we restrict the asymmetry to one firm, keeping the remaining symmetric. Thus, \( I \equiv \{1, 2, 3\} \), and let firm 1 be the asymmetric firm. Let \( m_2 = m_3 = m, r_2 = r_3 = r, c_2 = c_3 = c \). We simplify the notation by dropping the superscript notation and proceeding with the following notation for the linear value functions,

\[
\begin{align*}
V_1 &= \alpha_1 + \beta_1 x_1 + \gamma_1 x_2 + \gamma_1 x_3, \\
V_2 &= \alpha + \gamma x_1 + \beta x_2 + \eta x_3, \\
V_3 &= \alpha + \gamma x_1 + \eta x_2 + \beta x_3.
\end{align*}
\]  

We then apply Theorem 1. The advertising decisions are

\[
\begin{align*}
u_{1}^{*} &= \frac{\rho_1 \sqrt{1 - x_1}}{2c_1} (\beta_1 - \gamma_1), \\
u_{i}^{*} &= \frac{\rho \sqrt{1 - x_1}}{4c} (2\beta - \gamma - \eta), \forall i \neq 1.
\end{align*}
\]

For the 7 unknown parameters \( \alpha_1, \beta_1, \gamma_1, \alpha, \beta, \gamma \) and \( \eta \), seven equations are obtained:
\[
\begin{align*}
 r_i(\alpha_i + \beta_i + 2\gamma_i) &= m_i - \frac{2\delta}{3}(\beta_i + 2\gamma_i), \\
 r(\alpha + \beta + \gamma + \eta) &= m - \frac{2\delta}{3}(\beta + \gamma + \eta), \\
 (r_i + \delta)\beta_i &= m_i - \frac{\rho_i^2}{4c_i}(\beta_i - \gamma_i)^2, \\
 (r_i + \delta)\beta &= m - \frac{\rho_i^2}{16c}(2\beta - \gamma - \eta)^2, \\
 (r_i + \delta)\gamma_i &= -\frac{\rho_i^2}{8c}(2\beta - \gamma - \eta)(\gamma_i - \beta_i), \\
 (r_i + \delta)\gamma &= -\frac{\rho_i^2}{4c_i}(\beta - \gamma_i)(2\gamma - \beta - \eta), \\
 (r_i + \delta)\eta &= -\frac{\rho_i^2}{8c}(2\beta - \gamma - \eta)(2\eta - \beta - \eta).
\end{align*}
\]

(20)

For illustration, let \(m_i = 1.1\), \(m = 1\), \(\delta = 0.05\), \(r_i = 0.05, \forall i\), \(\rho_i^2 / c_i = 1, \forall i\). Solving these simultaneous equations using Maple 8, one obtains several solutions of which only the following solution is appropriate in the respect that the values are not imaginary and the value functions are always positive: \(\alpha_i = -3.3936\), \(\beta_i = 6.0197\), \(\gamma_i = 4.6083\), \(\alpha = -4.3268\), \(\beta = 5.7360\), \(\gamma = 4.4454\), \(\eta = 4.4147\).

### 4.2. Equilibrium market shares

Substituting the optimal advertising control expression from Theorem 1 into the original dynamics, we get

\[
\frac{dx_i}{dt} = \frac{n\rho_i^2(1-x_i)}{2c_i(n-1)^2} \left( n\phi_i - \sum_{j=1}^{n-1} \phi_j \right) - \sum_{j=1}^{n-1} \frac{\rho_j^2(1-x_j)}{2c_j(n-1)^2} \left( n\phi_j - \sum_{k=1}^{n-1} \phi_k \right) - \delta(x_i - \frac{1}{n}), \quad \forall i \in I.
\]

(21)

In equilibrium, \(\frac{dx_i}{dt} = 0\), and we can solve for the steady state market shares.
**Theorem 2**: Let (A1) and (A2) hold and let us define

\[ B_i \equiv \frac{\rho_i^2}{2c_i(n-1)^2}(n\phi_i' - \sum_{k=1}^n \phi_k') + \delta. \]

Then the following hold:

(a) The market share dynamics may be expressed as

\[ \frac{dx_i}{dt} = n(1-x_i)B_i - \sum_{j \neq i} (1-x_j)B_j, \quad \forall i \in I. \tag{22} \]

(b) The unique solution for the market share vector is \( x(t) = (I - e^{-Bt})I + e^B x(0) \), where

\[ B = \begin{pmatrix} -(n-1)B_1 & B_2 & \cdots & B_n \\ B_1 & -(n-1)B_2 & \cdots & B_n \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & -(n-1)B_n \end{pmatrix}. \tag{23} \]

(c) The equilibrium market share of the \( i \)th firm is given by the formula

\[ \bar{x}_i = 1 - \frac{n-1}{B \sum_{j=1}^n \frac{1}{B_j}}, \quad \forall i \in I. \tag{24} \]

Using this formula for a duopoly, the steady state market shares are given by the expression

\[ \bar{x}_i = \frac{B_i}{B_1 + B_2}, \quad i = 1, 2, \tag{25} \]

which is the result obtained in Prasad and Sethi (2003). For a triopoly, the expressions are

\[ \bar{x}_i = 1 - \frac{2B_iB_2B_3}{B_i(B_1B_2 + B_2B_3 + B_3B_1)}, \quad i = 1, 2, 3. \tag{26} \]
Apparently, the attraction model result of equilibrium market share being of the form 
“us/(us+them)”, which falls out from several Lanchester type dynamics, does not hold when 
there are more than two firms, that is, we did not obtain \( \bar{x}_i = B_i / \sum_{j=1}^i B_j \).

Note that if the \( B_i \)’s are equal for all firms, then each firm gets an equal \( 1/n \) market 
share, which is decreasing in the number of firms in the industry.

4.2.1 Illustration

We consider the simplified case of a four-firm oligopoly and use Theorem 2(a,b) to 
obtain market share trajectories and the steady state using numerical simulations. We then verify 
the steady state market shares against Theorem 2(c).

To obtain the trajectory, one may apply numerical methods of solving the system of 
linear differential equations in Theorem 2(a), such as the classical Runge-Kutta method. 
However, since we have a solution to the system of equations, we show how it may be directly 
applied. We begin by assuming that the \( B_i \) parameters for the industry are known, which may be 
from a study of historical data or expert judgment.

For illustration purposes, let \( B_1 = 1, B_2 = 1.1, B_3 = 1.2, B_4 = 1.5 \) and \( x(0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \). 
The market shares are given by \( y(t) = e^{At}y(0) \) where \( y(t) = 1 - x(t) \). It is computationally 
convenient to first diagonalize the matrix \( B \) as \( B = PAP^{-1} \), which yields (displayed to two 
significant digits)

\[
\begin{bmatrix}
-3 & 1.1 & 1.2 & 1.5 \\
1 & -3.3 & 1.2 & 1.5 \\
1 & 1.1 & -3.6 & 1.5 \\
1 & 1.1 & 1.2 & -4.5 \\
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
.73 & -.20 & -.20 & .74 \\
.66 & -.56 & -.26 & -.49 \\
.61 & .67 & -.39 & -.19 \\
.49 & .09 & .84 & -.06 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -4.62 & 0 & 0 \\
0 & 0 & -5.62 & 0 \\
0 & 0 & 0 & -4.16 \\
\end{bmatrix}
\begin{bmatrix}
.40 & .40 & .40 & .40 \\
-.21 & -.66 & .86 & .14 \\
-.14 & -.21 & -.34 & .93 \\
.86 & -.63 & -.26 & -.11 \\
\end{bmatrix}.
\]
Then, from the theory of matrices, \( y(t) = Pe^{A}P^{-1}y(0) \). Or, since \( x(t) = 1 - y(t) \),

\[
\begin{pmatrix}
  x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{pmatrix} =
\begin{pmatrix}
  .73 & -.20 & -.20 & .74 \\
  .66 & -.56 & -.26 & -.49 \\
  .61 & .67 & -.39 & -.19 \\
  .49 & .09 & .84 & -.06
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & e^{-4.62t} & 0 & 0 \\
  0 & 0 & e^{-5.62t} & 0 \\
  0 & 0 & 0 & e^{-4.16t}
\end{pmatrix}
\begin{pmatrix}
  .40 & .40 & .40 & .40 \\
  -.21 & -.66 & .86 & .14 \\
  -.14 & -.21 & -.34 & .93 \\
  .86 & -.63 & -.26 & -.11
\end{pmatrix}
\begin{pmatrix}
  .75 \\
  .75 \\
  .75 \\
  .75
\end{pmatrix}.
\] (28)

This simplifies to

\[
\begin{pmatrix}
  x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{pmatrix} =
\begin{pmatrix}
  0.12000 + 0.01935e^{-4.61776t} + 0.03444e^{-5.62262t} + 0.07621e^{-4.15962t} \\
  0.20000 + 0.05490e^{-4.61776t} + 0.04571e^{-5.62262t} - 0.05060e^{-4.15962t} \\
  0.26667 - 0.06560e^{-4.61776t} + 0.06793e^{-5.62262t} - 0.01900e^{-4.15962t} \\
  0.41333 - 0.00865e^{-4.61776t} - 0.14807e^{-5.62262t} - 0.00661e^{-4.15962t}
\end{pmatrix}.
\] (29)

The results are plotted in Figure 1.

<Insert Figure 1 here>

Using the formula in Theorem 2(c), the steady state market shares are \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = (0.12, 0.20, 0.267, 0.413)\). Equation (29) says that the optimal solution asymptotically approaches the equilibrium market share.

4.3. Applicability

We now consider the implications of Assumptions (A1) and (A2) on the applicability of the results. Assumption (A1) requires that the firm should commit strictly positive advertising at all times as its optimal strategy. If the expression for \( u_i^* \) is positive, then it would appear that this condition has been met. In particular, we can verify numerically that this condition is met given the parameters for the market. Those parameters may in turn be determined by managerial judgment or through estimation on historical advertising-sales data. In the case of symmetric firms, we can verify this condition analytically, which we shall do in the next section.
However, it is not sufficient to show only that the expression for $u_i^*$ is positive. One must also calculate the value function for the firm and show that it is positive as well. Else, the firm is guaranteed to make a strictly positive profit by having zero advertising expenditure. We shall also examine this condition analytically for symmetric firms.

By Assumption (A2) stated above, the model results will apply to industries where the industry parameters are such that the market share for no firm should be less than 0 or greater than 1 in equilibrium. We derive the sufficient condition for this applicability condition in this section. From the previous section, the dynamics along an optimal path are given by

$$\frac{dx_i}{dt} = n(1-x_i)B_i - \sum_{j=1}^{n} (1-x_j)B_j, \quad \forall i \in I. \quad (30)$$

For the market share of firm $i$ to be less than 1, the required condition is that $\frac{dx_i}{dt} \leq 0$ when $x_i = 1$. An examination of (30) shows that this condition is always satisfied.

For the market share to be greater than 0, the required condition is that $\frac{dx_i}{dt} \geq 0$ when $x_i = 0$. This condition is always satisfied if $n = 2$ since $x_j = 1$ and $\frac{dx_i}{dt} = nB_i$. Let us, however, consider the case when $n > 2$. For this analysis, let us renumber the firms such that $B_1 \leq B_2 \leq \ldots \leq B_n$. We have the following result:

**Theorem 3:** The requirement that

$$B_i \geq \frac{1}{(n-1)} \sum_{j=3}^{n} B_j \quad (31)$$

is a sufficient condition for Assumption (A2) to hold.
Thus, in the three and four firm cases, $B_i \geq B_i/2$ and $B_i \geq (B_i + B_i)/3$ are the required conditions, respectively. Let us now consider the extreme case of market structure where $B_3 = B_4 = \ldots = B_n = B$. Then (31) gives $B_i \geq \frac{(n-2)B}{n-1}$ as the required condition. Since $\lim_{n \to \infty} \frac{n-2}{n-1} = 1$, we get $B_i \geq B$. But by definition $B_i \leq B_i$, thus $B_i = B$ is the required condition in the limit as the number of firm becomes very large. This allows a progressively decreasing spread between the highest and lowest valuation firms as the number of firms increases. Hence, we cannot study extremely asymmetric markets. However, with symmetric firms, the requirement is always satisfied.

Applicability is increased if we do not enforce that market shares should always be non-negative, but instead adopt the less stringent condition that steady state market shares should be non-negative. That condition is given by

$$B_i \sum_{j=1}^{n} \frac{1}{B_j} \geq n-1. \quad (32)$$

It can be verified that if (31) holds, then (32) must also hold.

### 4.4. Symmetric firms

We will solve for the unknown parameters in the case when firms are symmetric, i.e.,

$\forall i \in I, \ m_i = m, \ \rho_i = \rho, \ \gamma_i = \gamma$ and, thus, $\phi_i^0 = \alpha, \ \phi_i^j = \beta_i, \ \phi_i^j (\forall j \neq i) = \gamma$. By construction, there is a unique and explicit solution to the differential game or a symmetric oligopoly market. By symmetry $B_i = B, \forall i$. Hence, the dynamics simplifies considerably to

$$\frac{dx_i}{dt} = nB(\frac{1}{n} - x_i), \quad \forall i \in I. \quad (33)$$
This shows that each firm moves monotonically from its initial market share to the steady state market share of $1/n$. Further results are given in Theorem 4.

**Theorem 4**: For $n$ symmetric firms,

(a) the value function is $V_i = (\alpha + \gamma) + (\beta - \gamma)x_i$, where

$$\beta - \gamma = \frac{2m}{\sqrt{(r + \delta)^2 + \frac{m\rho^2(n + 1)}{c(n - 1)}} + (r + \delta)} > 0; \quad (34)$$

(b) The advertising control for each firm is $u^* = \frac{\rho\sqrt{1 - x}}{2c}(\beta - \gamma)$;

(c) $\alpha = \frac{m}{r} \cdot \frac{r + \delta - \delta/n}{r(n + 1)} \cdot \left( \frac{2mn}{r + \delta} \cdot (n - 1)(\beta - \gamma) \right)$, $\beta = \frac{1}{n + 1} \cdot \left( \frac{2m}{r + \delta} + (n - 1)(\beta - \gamma) \right)$, $\gamma = \frac{2}{n + 1} \cdot \left( \frac{m}{r + \delta} - (\beta - \gamma) \right)$, and $B = \frac{\rho^2(\beta - \gamma)}{2c(n - 1)} + \delta/n$.

The expression for $n = 2$ corresponds to the symmetric duopoly solution of Prasad and Sethi (2003), where the $\beta$, $c$ and $\delta$ parameters in that paper correspond to $(\beta - \gamma)$, $c$ and $\delta/2$, respectively, in this paper. From direct observation of Theorem 4(b, c), the following comparative statics can be noted:

$$\frac{\partial(\beta - \gamma)}{\partial m} > 0, \frac{\partial(\beta - \gamma)}{\partial r} < 0, \frac{\partial(\beta - \gamma)}{\partial \delta} < 0, \frac{\partial(\beta - \gamma)}{\partial \rho} < 0, \frac{\partial(\beta - \gamma)}{\partial c} > 0, \frac{\partial(\beta - \gamma)}{\partial n} > 0. \quad (35)$$

$$\frac{\partial u^*}{\partial m} > 0, \frac{\partial u^*}{\partial r} < 0, \frac{\partial u^*}{\partial \delta} < 0, \frac{\partial u^*}{\partial \rho} > 0, \frac{\partial u^*}{\partial c} < 0, \frac{\partial u^*}{\partial n} > 0. \quad (36)$$

Note in particular that the firm’s advertising is increasing with the number of firms in the industry. While this is in contrast to the result of Fershtman (1984), the results for the interest
rate, depreciation rate and production cost (captured by the margin) are consistent. The total industry advertising \( nu^* \) is also increasing with the number of firms. This leads to the consideration of whether Assumption (A1) will hold at all times or not. One can do the calculations in two ways. The first is to calculate \( \alpha + \gamma \) and examine the value function for a firm to see whether it becomes negative. We find that

\[
\alpha + \gamma = \frac{m}{r} + \frac{n-1}{r(n+1)} \left( -2m + (\beta - \gamma)(r + \delta - \delta / n - \frac{2r}{n-1}) \right), \tag{37}
\]

and it may be seen that this is decreasing in the number of firms, and further that it may become negative for large enough \( n \). One may interpret an upper bound on the number of firms in the following manner. Suppose that firms enter the industry one at a time and that an entering firm has zero market share initially. Its value function is then zero if it does no advertising. But if it follows the advertising rule in Theorem 4, which states that \( u^* = \frac{\rho \sqrt{1-x}}{2c} (\beta - \gamma) \), then its value function is just \( \alpha + \gamma \) (recall from Theorem 4 that the value function when market share is \( x_i \) is given by \( V_i = (\alpha + \gamma) + (\beta - \gamma)x_i \)). The required condition for entry then is that \( \alpha + \gamma > 0 \).

The second calculation is simpler. We write the total industry profit as

\[
\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \left( (\alpha + \gamma) + (\beta - \gamma)x_i \right) = n(\alpha + \gamma) + (\beta - \gamma). \tag{38}
\]

We have shown that \( \beta - \gamma > 0 \), but the first term can become negative with an increase in the number of firms showing that the industry can sustain only a limited number \( n^* \) of firms before zero advertising becomes an optimal strategy for at least a subset of firms. The sustainable number is determined by setting

\[
n(\alpha + \gamma) + (\beta - \gamma) \geq 0. \tag{39}
\]
This condition is less stringent than the requirement that an individual firm’s value function be positive, but it has the benefit of applying to an existing configuration of firms rather than the earlier sequential entry description.

We rephrase this condition as

$$\sum_{i=1}^{\infty} V_i = \int_0^\infty e^{-\eta}(m-c) \sum_{i=1}^{\infty} u_i^*(t)^2 dt = \frac{m}{r} \frac{\rho^2(\beta-\gamma)^2(n-1)}{4cr} > 0$$

(40)

We insert the expression for $\beta - \gamma$ from Theorem 4 into this, and with some algebraic manipulation obtain the condition

$$\frac{m\rho^2n^*(3-n^*)}{n^*-1} + 2c(r+\delta)(\sqrt{(r+\delta)^2 + \frac{m\rho^2(n^*+1)}{c(n^*-1)}} + (r+\delta)) \geq 0$$

$$\Rightarrow n^*(3-n^*) + \frac{2(n^*+1)}{\sqrt{1 + \frac{m\rho^2(n^*+1)}{(r+\delta)^2 c(n^*-1)} - 1}} \geq 0.$$  

(41)

This is clearly satisfied for duopoly and triopoly, but with more firms, the first term becomes negative, while the second is always positive. Hence, numerical investigation is required. Figure 2 shows the plot of the function in equation (41) for different values of a parameter $K \equiv m\rho^2/(r+\delta)^2 c$.

<Insert Figure 2 here>

If it is the case that we start with the number of firms in the industry that is larger than the sustainable number, then a shakeout will occur. Some firms will do no advertising and allow their market share to decline to zero thereby harvesting the profits from their initial market share. Presumably, when the market share has declined to zero, these firms will exit the market, thus reducing the number of firms until it is at or below the sustainable number. Hence, in practice, the constraint that the number of firms should be below $n^*$ does not appear to be a serious one.
5. Conclusions

The annual advertising expenditure is over $100 billion for firms in the US alone, but it is often spent in a suboptimal manner (Aaker and Carmen 1982). Analysis of optimal advertising expenditures in an oligopoly market is important, but few attempts have been made in this regard due to the complexity of the modeling. We employ a fruitful avenue of research based on the Sethi model that allows for greater tractability in analysis, due to a linear form of the value function that results from using it. By doing so, we extend the duopoly analysis of Sorger (1989) and Sethi and Prasad (2003) to \( n \)-firms.

In the general case, and subject to two assumptions on the domain of applicability of the model, we obtain the closed-loop equilibrium advertising strategies, and show that advertising should be proportional to the combined competitors’ market shares. The market shares in equilibrium are given by a simple expression based on the strengths of the different brands.

In the case of a symmetric oligopoly market, we are able to obtain an explicit closed-loop solution for the advertising expenditures of all firms. The comparative statics in this case agree with the results of Fershtman (1984) in a goodwill based model of advertising in oligopoly markets, namely, if all firms are identical, the firms’ equilibrium advertising decreases with interest rate and production cost increases. We are also in agreement with Erickson’s (2003) result for triopoly markets, namely, advertising increases with the profit margin. In contrast with Fershtman, we find that advertising increases as the number of firms increases. As a consequence, we also find that the value function decreases with an increase in the number of firms. However, there is a limit to the number of firms with positive advertising that can be accommodated by the industry.
Additional research may consider the possibility of including stochasticity into the model as in Horsky and Mate (1988) and Prasad and Sethi (2003). The difficulty there will be the intractability caused by constraining the market shares to lie between 0 and 1. Sales expansion effects of advertising models such as this have also been an area of investigation; see, e.g., Erickson (1985) and Fruchter (1999). Finally, the results obtained here could be tested using industry data.
Appendix A

Proof of Theorem 1

The Hamilton-Jacobi-Bellman (HJB) equation for firm $i$ is given by

$$rV_i = \max_{\mu_i} m_i x_i - c_i \mu_i^2 + \sum_{j \in I} \frac{\partial V_i}{\partial x_j} \left( \frac{n}{n-1} \rho_j \mu_j \sqrt{1-x_j} - \frac{1}{n-1} \sum_{k \in I} \rho_k \mu_k \sqrt{1-x_k} - \delta(x_j - \frac{1}{n}) \right).$$

(A1)

This may be rewritten as

$$rV_i = \max_{\mu_i} m_i x_i - c_i \mu_i^2 + \sum_{j \in I} \frac{\rho_j \mu_j \sqrt{1-x_j}}{n-1} \left( n \frac{\partial V_i}{\partial x_j} - \sum_{k \in I} \frac{\partial V_i}{\partial x_k} \right) - \sum_{j \in I} \frac{\partial V_i}{\partial x_j} \delta(x_j - \frac{1}{n}).$$

(A2)

We obtain the optimal feedback controls

$$u_i^* = \max\{0, \frac{\rho_i \sqrt{1-x_i}}{2c_i (n-1)} \left( n \frac{\partial V_i}{\partial x_i} - \sum_{k \in I} \frac{\partial V_i}{\partial x_k} \right) \}, \quad \forall i \in I.$$

(A3)

Anticipating that the controls will be shown to be positive, we insert these controls into the HJB equations to obtain for firm $i$,

$$rV_i = m_i x_i - \frac{\rho_i^2 (1-x_i)}{4c_i (n-1)^2} \left( n \frac{\partial V_i}{\partial x_i} - \sum_{k \in I} \frac{\partial V_i}{\partial x_k} \right)^2$$

$$+ \sum_{j \in I} \frac{\rho_j^2 (1-x_j)}{2c_j (n-1)^2} \left( n \frac{\partial V_i}{\partial x_j} - \sum_{k \in I} \frac{\partial V_i}{\partial x_k} \right) \left( n \frac{\partial V_i}{\partial x_j} - \sum_{k \in I} \frac{\partial V_i}{\partial x_k} \right) - \sum_{j \in I} \frac{\partial V_i}{\partial x_j} \delta(x_j - \frac{1}{n}).$$

(A4)

To solve these $n$ simultaneous partial differential equations, we use the following linear value functions, observing that they satisfy the Hamilton-Jacobi equations,

$$V_i = \phi_0 + \sum_{j=1}^n \phi_j x_j, \quad \forall i \in I.$$

(A5)

Thus, there are a total of $n+1$ unknown parameters for each of the $n$ firms. To determine these parameters, we insert equation (A5) into the Hamilton-Jacobi equation and obtain $\forall i \in I$, 

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\[ r_i(\phi_i' + \sum_{j=1}^{n} \phi_j x_j) = m_i x_i - \frac{\rho_i^2 (1-x_i)^2}{4 c_i (n-1)^2} \left( n\phi_i' - \sum_{k=1}^{n} \phi_k' \right)^2 \]

\[ + \sum_{j=1}^{n} \frac{\rho_j^2 (1-x_j)^2}{2 c_j (n-1)^2} \left( n\phi_j' - \sum_{k=1}^{n} \phi_k' \right) \left( n\phi_j' - \sum_{k=1}^{n} \phi_k' \right) - \sum_{j=1}^{n} \delta \phi_j x_j + \frac{\delta}{n} \sum_{j=1}^{n} \phi_j. \]  

(A6)

\[ \Rightarrow r_i \left( \sum_{j=0}^{n} \phi_j' - \sum_{j=1}^{n} \phi_j (1-x_j) \right) = m_i - m_i (1-x_i) - \frac{\rho_i^2 (1-x_i)^2}{4 c_i (n-1)^2} \left( n\phi_i' - \sum_{k=1}^{n} \phi_k' \right)^2 \]

\[ + \sum_{j=1}^{n} \frac{\rho_j^2 (1-x_j)^2}{2 c_j (n-1)^2} \left( n\phi_j' - \sum_{k=1}^{n} \phi_k' \right) \left( n\phi_j' - \sum_{k=1}^{n} \phi_k' \right) + \delta \sum_{j=1}^{n} \phi_j (1-x_j) - \frac{(n-1)\delta}{n} \sum_{j=1}^{n} \phi_j. \]  

(A7)

Equating powers of \((1-x_i)\), we get

\[ r_i \left( \sum_{j=0}^{n} \phi_j' \right) = m_i - \frac{(n-1)\delta}{n} \sum_{j=1}^{n} \phi_j, \]  

(A8)

\[ (r_i + \delta)\phi_i' = m_i - \frac{\rho_i^2}{4 c_i (n-1)^2} \left( n\phi_i' - \sum_{k=1}^{n} \phi_k' \right)^2, \]  

(A9)

\[ (r_i + \delta)\phi_j' = -\frac{\rho_j^2}{2 c_j (n-1)^2} \left( n\phi_j' - \sum_{k=1}^{n} \phi_k' \right) \left( n\phi_j' - \sum_{k=1}^{n} \phi_k' \right), \forall j \in I, j \neq i. \]  

(A10)

These \(n+1\) relationships are obtained \(\forall i \in I\), resulting in \(n(n+1)\) equations to be solved for the \(n(n+1)\) unknown coefficients.

**Proof of Theorem 2**

Part (a): We can write equation (21) as

\[ \frac{dx_i}{dt} = n(1-x_i)A_i - \sum_{j=1}^{n} (1-x_j)A_j - \delta(x_i - \frac{1}{n}), \quad \forall i \in I, \]  

(B1)

where we defined \(A_i = \frac{\rho_i^2}{2 c_i (n-1)^2} \left( n\phi_i' - \sum_{k=1}^{n} \phi_k' \right)\). This may be further simplified in the following manner:
\[
\frac{dx_i}{dt} = n(1 - x_i)A_i - \sum_{j=1}^{n} (1 - x_j)A_j - \delta(x_i - 1) - \frac{\delta}{n} \sum_{j=1}^{n} (1 - x_j) \\
\Rightarrow \frac{dx_i}{dt} = n(1 - x_i)(A_i + \frac{\delta}{n}) - \sum_{j=1}^{n} (1 - x_j)(A_j + \frac{\delta}{n}) \\
\Rightarrow \frac{dx_i}{dt} = n(1 - x_i)B_i - \sum_{j=1}^{n} (1 - x_j)B_j
\]

where \( B_i \equiv A_i + \frac{\delta}{n} = \frac{\rho_i^2}{2c_i(n-1)^2} \left( n\phi_i - \sum_{k=1}^n \phi_k \right) + \frac{\delta}{n} \).

Part (b): The dynamics may be further written as

\[
\begin{pmatrix}
\frac{dy_1}{dt} \\
\frac{dy_2}{dt} \\
\vdots \\
\frac{dy_n}{dt}
\end{pmatrix} =
\begin{pmatrix}
-(n-1)B_1 & B_2 & \cdots & B_n \\
B_1 & -(n-1)B_2 & \cdots & B_n \\
\vdots & \vdots & \ddots & \vdots \\
B_1 & B_2 & \cdots & -(n-1)B_n
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix},
\]

where \( y_i \equiv 1 - x_i \). Or, as

\[
\frac{dy(t)}{dt} = By(t),
\]

where the variables are defined by direct correspondence to (B3). The unique solution of this system of equations is \( y(t) = e^{B}y(0) \), or

\[
x(t) = (I - e^{-B})I + e^{B}x(0),
\]

where \( I \) is the identity matrix, \( 1 \) is a column vector of 1’s, and

\[
e^{B} = I + tB + \frac{t^2B^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(tB)^k}{k!}.
\]

Part (c): In equilibrium, \( \frac{dx_i}{dt} = 0 \) and we can solve for the steady state market shares. Let \( \pi_i \) denote the market share in steady state for firm \( i \). From (B2),
\[ 1 - \bar{x}_i = \frac{1}{nB_i} \sum_{j \in I} (1 - \bar{x}_j)B_j \]  \hspace{1cm} (B6)

\[ \Rightarrow \sum_{i=1}^{n} (1 - \bar{x}_i) = \sum_{i=1}^{n} \frac{1}{nB_i} \sum_{j \in I} (1 - \bar{x}_j)B_j \]

\[ \Rightarrow \frac{n-1}{B_i} = \frac{1}{n} \sum_{j \in I} (1 - \bar{x}_j)B_j \]

\[ \Rightarrow \frac{n-1}{B_i} \sum_{j \in I} \frac{1}{B_j} = \frac{1}{nB_i} \sum_{j \in I} (1 - \bar{x}_j)B_j \]

And inserting the last expression back into (B6) gives us the desired result

\[ \bar{x}_i = 1 - \frac{n-1}{B_i} \sum_{j \in I} \frac{1}{B_j}, \quad \forall i \in I. \]  \hspace{1cm} (B7)

**Proof of Theorem 3**

We require that \( \frac{dx_i}{dt} \geq 0 \), when \( x_i = 0 \). Inserting this into (30), the required condition may be restated as

\[ nB_i \geq \sum_{j \in I} (1 - x_j)B_j, \quad \forall i \in I. \]  \hspace{1cm} (B8)

To obtain a sufficient condition, we will consider the most pessimistic condition. Renumber the firms such that \( B_1 \leq B_2 \leq \ldots \leq B_n \). Then, we pick the smallest value of the left hand side and the largest value of the right hand side. This gives the required condition

\[ nB_1 \geq \sum_{j=1}^{x_1} B_j - B_2, \quad \text{or} \]

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Proof of Theorem 4

From Theorem 1 we have the relationships
\[
(r + \delta) \alpha + (r + \delta - \frac{\delta}{n})(\beta + (n-1)\gamma) = m, \tag{C1}
\]
\[
(r + \delta) \beta = m - \frac{\rho^2(\beta - \gamma)^2}{4c}, \tag{C2}
\]
\[
(r + \delta) \gamma = \frac{\rho^2(\beta - \gamma)^2}{2c(n-1)}. \tag{C3}
\]
Subtracting (C3) from (C2), one obtains a quadratic equation in \(\beta - \gamma\), which yields
\[
\beta - \gamma = \left( \pm \sqrt{(r + \delta)^2 + \frac{m\rho^2(n+1)}{c(n-1)} - (r + \delta) \left( \frac{\rho^2(n+1)}{2c(n-1)} \right)} \right). \tag{C4}
\]
Note that \(V_i = \alpha + \beta x_i + \sum_{j \neq i} \gamma x_j\) can be written as \(V_i = (\alpha + \gamma) + (\beta - \gamma)x_i\) after using
\[
\sum_{j \neq i} x_j = 1 - x_i. \]
Hence, we expect \(\beta - \gamma > 0\), and only the positive root should be taken. The expression can be further simplified to
\[
\beta - \gamma = \frac{2m}{\sqrt{(r + \delta)^2 + \frac{m\rho^2(n+1)}{c(n-1)} + (r + \delta)}}. \tag{C5}
\]
We express the unknowns in terms of \(\beta - \gamma\). Multiplying (C2) by 2, and (C3) by \(n-1\) and adding them, we get

\[(n-1)B_i \geq \sum_{j=3}^{n} B_j. \tag{B9}\]
\[ 2\beta + (n-1)\gamma = 2m/(r+\delta) \]

\[ \Rightarrow \gamma = \frac{2}{n+1} \left( \frac{m}{r+\delta} - (\beta - \gamma) \right) \]  

\[ \Rightarrow \beta = \frac{1}{n+1} \left( \frac{2m}{r+\delta} + (n-1)(\beta - \gamma) \right) \]  

Substituting this in (C1),

\[ \alpha = \frac{m}{r} - \frac{r+\delta - \delta/n}{r(n+1)} \left( \frac{2mn}{r+\delta} - (n-1)(\beta - \gamma) \right) \]  

\[ \text{(C7)} \]
References


Table 1: List of variables and parameters

<table>
<thead>
<tr>
<th>Notation</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i(t) \in [0,1] )</td>
<td>Market share of firm ( i ), ( i \in 1 \equiv {1,2,\ldots,n} ) at time ( t ).</td>
</tr>
<tr>
<td>( u_i(x(t),t) \geq 0 )</td>
<td>Advertising rate by firm ( i ) at time ( t ), where ( x(t) \equiv (x_1(t),x_2(t),\ldots,x_n(t))' ).</td>
</tr>
<tr>
<td>( \rho_i &gt; 0 )</td>
<td>Advertising effectiveness parameter for firm ( i ).</td>
</tr>
<tr>
<td>( \xi_i &gt; 0 )</td>
<td>Competitive advertising decay parameter.</td>
</tr>
<tr>
<td>( \delta &gt; 0 )</td>
<td>Churn parameter.</td>
</tr>
<tr>
<td>( r_i &gt; 0 )</td>
<td>Discount rate for firm ( i ).</td>
</tr>
<tr>
<td>( C(u_i(t)) )</td>
<td>Cost of advertising, parameterized as ( c_i u_i(t)^2 ), ( c_i &gt; 0 ).</td>
</tr>
<tr>
<td>( m_i &gt; 0 )</td>
<td>Industry sales multiplied by the per unit profit margin for firm ( i ).</td>
</tr>
<tr>
<td>( V_i )</td>
<td>Value function for firm ( i ).</td>
</tr>
<tr>
<td>( \alpha, \beta, \gamma, \phi )</td>
<td>Components of the value function.</td>
</tr>
<tr>
<td>( B_i )</td>
<td>A useful intermediate term, ( B_i \equiv \rho_i^{-2}(n\phi_i^2 - \sum_{k \neq i} \phi_k^2) / 2c_i(n-1)^2 + \delta / n ).</td>
</tr>
</tbody>
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Fig. 1. Market Share Paths in a 4-Firm Oligopoly

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**Fig. 1. Market Share Paths in a 4-Firm Oligopoly**

- $X_1(t)$
- $X_2(t)$
- $X_3(t)$
- $X_4(t)$
Fig. 2. Maximum sustainable number of symmetric firms

- $F(n^*, K=0.5)$
- $F(n^*, K=1)$
- $F(n^*, K=10)$
- $F(n^*, K=100)$