Continous-Time Fourier Transform

Fourier series represents periodic signals as linear combinations of complex exponentials. Here we extend this concept to apply to signals that are not periodic. Aperiodic signals with finite energy can also be represented through a linear combination of complex exponentials.

<table>
<thead>
<tr>
<th>Building Blocks</th>
<th>Periodic signals</th>
<th>Aperiodic signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complex Exponentials combination</td>
<td>Horizontally Related sum</td>
<td>very close in frequency Integral</td>
</tr>
</tbody>
</table>
Development of the Fourier Transform

Revisit the Fourier series representation of the continuous-time periodic square wave.

Note that the pulse $x(t)$ is equal to the pulse train in the limit as $T \to \infty$; that is in mathematical terms
Development of the Fourier Transform

\[ x(t) = \lim_{T \to \infty} x_T(t) \]

Fourier Series representation of \( X_T[t] \) is

\[ x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

where \( \omega_0 = \frac{2\pi}{T} \)

\[ a_k = \frac{2 \sin(k\omega_0 T)}{k\omega_0 T} \]

Since \( x(t) = x_T(t) \) for \( |t| < T/2 \) and since \( x(t) = 0 \) outside this interval.
Development of the Fourier Transform

For the coefficients \( a_k \)

\[
a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-j\omega t} \, dt = \frac{1}{T} \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt
\]

Define the envelope \( X(j\omega) \) of \( Ta_k \) as (Fourier Transform of \( x(t) \))

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt
\]
Development of the Fourier Transform

\[
x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} \overline{X}(jk\omega_0) e^{jk\omega_0 t} \]

\[
\omega_0 = \frac{2\pi}{T}
\]

\[
x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{T} \overline{X}(jk\omega_0) e^{jk\omega_0 t} \omega_0
\]

as \( T \to \infty \), \( \omega_0 \to 0 \) and the right-hand side of equation passes to an integral.

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t} \, dt \to \text{Inverse Fourier Transform Equation}
\]

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \to \text{Fourier Transform Equation}
\]
Convergence of Fourier Transforms

Dirichlet Conditions:

1. $x[t]$ be absolutely integrable; that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

2. Finite number of minima and maxima

3. Finite number of discontinuities.
Examples of Continuous-time Fourier Transform

Consider the signal

\[ X[t] = e^{-at} u(t) \]

\[
X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = \frac{1}{a+j\omega} e^{-(a+j\omega)t} \bigg|_0^\infty
\]

\[ X(j\omega) = \frac{1}{a+j\omega}, a > 0 \]

Since this Fourier Transform is complex valued, to plot it as a function of \( \omega \), we express \( X(j\omega) \) in terms of its magnitude and phase.

\[
|X(j\omega)| = \frac{1}{\sqrt{a^2 + b^2}} \quad \text{phase}\{X(j\omega)\} = -\tan^{-1}\left(\frac{\omega}{a}\right)
\]
Example contd:
Example:

Let \( x(t) = e^{-a|t|}, a > 0 \)

The Fourier Transform of the signal is

\[
X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt
\]

\[
= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt
\]

\[
= \frac{1}{a+j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2 + \omega^2}
\]