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Supplementary Appendix

1. Computation of Outward Normal

In this section, we will discuss on the computation of the outward normal $\mathbf{n}_{ij}$ of medial primitive $\mathbf{P}_j$ at the surface vertex $\mathbf{v}_i$.

If the medial primitive $\mathbf{P}_j$ is a medial cone $\mathbf{e}_{kl}$, we denote $\mathbf{d}_c$ to be the center direction form $\mathbf{c}_k$ to $\mathbf{c}_l$, that is:

$$\mathbf{d}_c = (\mathbf{c}_l - \mathbf{c}_k)/\|\mathbf{c}_l - \mathbf{c}_k\|, \tag{1}$$

and d_r to be the radius gradient from $\mathbf{m}_k$ to $\mathbf{m}_l$, that is:

$$d_r = (r_l - r_k)/\|\mathbf{c}_l - \mathbf{c}_k\|. \tag{2}$$

As shown in Figure 1, $\mathbf{v}_i'$ is the perpendicular projection of $\mathbf{v}_i$ on line $\overline{\mathbf{c}_k\mathbf{c}_l}$ (the orange point in Figure 1), then the outward normal of $\mathbf{P}_j$ at $\mathbf{v}_i$ is a unit vector $\mathbf{n}_{ij}$ that satisfies:

$$\mathbf{n}_{ij} \cdot \mathbf{d}_c = -d_r. \tag{3}$$

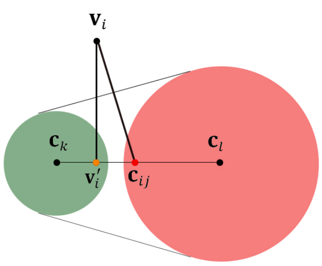

Figure 1: *The illustration of outward normal of a cone at surface vertex.*

Without loss of generality, let us consider that $\mathbf{v}_i$ is not on the line $\overline{\mathbf{c}_k\mathbf{c}_l}$, and it can be represented as follows:

$$\mathbf{n}_{ij} = \alpha \frac{(\mathbf{v}_i' - \mathbf{v}_i)}{\|\mathbf{v}_i' - \mathbf{v}_i\|} + \beta \mathbf{d}_c, \tag{4}$$

where α and β are scalars to be determined. Combining Eq. (3) and Eq. (4) with a unit vector condition for $\mathbf{n}_{ij}$, we could solve $\mathbf{n}_{ij}$ as follows:

$$\mathbf{n}_{ij} = \frac{(\mathbf{v}_i' - \mathbf{v}_i)}{\|\mathbf{v}_i' - \mathbf{v}_i\|} \sqrt{1 - {d_r}^2} - \mathbf{d}_c \cdot d_r. \tag{5}$$

Then a footprint $\mathbf{m}_{ij} = (\mathbf{c}_{ij}, r_{ij})$ of $\mathbf{v}_i$ on $\mathbf{P}_j$ could be computed by this outward normal $\mathbf{n}_{ij}$. Please refer Sec. 3.1 for the computation of $\mathbf{m}_{ij}$. If the footprint is an inner-footprint, we take the computed vector as the outward normal of $\mathbf{P}_j$ at $\mathbf{v}_i$. If the footprint $\mathbf{m}_{ij}$ is an outer-footprint and getting clamped to the boundary sphere of either $\mathbf{m}_k$ or $\mathbf{m}_l$, the outward normal could be computed as follows:

$$\mathbf{n}_{ij} = \frac{\mathbf{v}_i - \mathbf{c}_{ij}}{\|\mathbf{v}_i - \mathbf{c}_{ij}\|}. \tag{6}$$

If the medial primitive $\mathbf{P}_j$ is a medial slab $\mathbf{f}_{klt}$, we denote $\mathbf{n}_j^g, g \in \{1,2\}$ to be the normals of the two triangles bounding the medial slab. The outward normal can be selected by the minimizer of the following absolute distance $F_d(\mathbf{n})$:

$$F_d(\mathbf{n}) = \|(\mathbf{v}_i - \mathbf{c}_k)\cdot \mathbf{n}\|, \quad \mathbf{n} \in \left\{\mathbf{n}_j^g | g = 1,2\right\}, \tag{7}$$

where $\mathbf{c}_k$ is the center of any one medial sphere $\mathbf{m}_k$ on the triangle of the slab. Then we will use the selected normal from $\{\mathbf{n}_j^g | g = 1,2\}$ for computing a footprint $\mathbf{m}_{ij}$ of $\mathbf{v}_i$ on $\mathbf{f}_{klt}$.

If the footprint is an inner-footprint on $\mathbf{f}_{klt}$, we take the selected normal as the outward normal at $\mathbf{v}_i$. Otherwise, we will compute an outer-footprint on the boundary cones as follows: we firstly compute the outward normals of the three boundary cones $\mathbf{e}_{kl}$, $\mathbf{e}_{lt}$, $\mathbf{e}_{kt}$ at the vertex $\mathbf{v}_i$, and select the inner-footprint with smallest squared distance E_d defined in Sec. 3.1 and the corresponding outward normal for $\mathbf{v}_i$. If there is no inner-footprint on all of the three boundary cones, we will select from the three medial spheres $\mathbf{m}_k$, $\mathbf{m}_l$, $\mathbf{m}_t$ with the smallest E_d and use the formula in Eq. (6).

2. Optimization of Rigid Motion

In this section, we will discuss on the optimization of the rigid motion $\{\mathbf{R}_j, \mathbf{t}_j\}$ of medial primitive $\mathbf{P}_j$ in the following total energy E:

$$E\left(\{\mathbf{R}_j, \mathbf{t}_j\}, \{\mathbf{m}_l\}\right) = \sum_j (Q_j + \omega W_j). \tag{8}$$

In our ARAP total energy E of Eq. (8), only W_j involves the translation $\mathbf{t}_j$ of medial primitive $\mathbf{P}_j$, thus the optimal translation $\mathbf{t}_j$ could be computed by:

$$\frac{\partial W_j}{\partial \mathbf{t}_j} = \mathbf{0} \Leftrightarrow \sum_{\mathbf{m}_l \in \mathcal{V}_j} \left(\mathbf{R}_j \mathbf{c}_l^0 + \mathbf{t}_j - \mathbf{c}_l\right) = \mathbf{0}. \tag{9}$$

To make the optimal translation $\mathbf{t}_j$ independent with $\mathbf{R}_j$, we set that each medial primitive has its local coordinate system with origin on the center of the primitive. Then it satisfies the following condition:

$$\sum_{\mathbf{m}_l \in \mathcal{V}_j} \left(\mathbf{R}_j \mathbf{c}_l^0\right) = \mathbf{0}, \tag{10}$$

and the optimal translation $\mathbf{t}_j$ at transformed frame could be computed as:

$$\mathbf{t}_j = \frac{\sum_{\mathbf{m}_l \in \mathcal{V}_j} \mathbf{c}_l}{|\mathcal{V}_j|}. \tag{11}$$

In the total energy E of Eq. (8), the rotation $\mathbf{R}_j$ of each medial primitive $\mathbf{P}_j$ only depends on its corresponding terms $Q_j + \omega W_j$, which can be expanded as:

$$\begin{aligned} Q_j + \omega W_j = &\sum_{\mathbf{v}_i \in \mathcal{C}_j} \left(A - 2\left(\mathbf{R}_j \mathbf{u}_{ij} r_{ij}\right)^\top (\mathbf{v}_i - \mathbf{c}_{ij})\right) \\ &+ \omega \sum_{\mathbf{m}_l \in \mathcal{V}_j} \left(B - 2\left(\mathbf{R}_j \mathbf{c}_l^0\right)^\top (\mathbf{c}_l - \mathbf{t}_j)\right), \end{aligned} \tag{12}$$

where:

$$\begin{aligned} A &= \left(\mathbf{u}_{ij} r_{ij}\right)^\top \left(\mathbf{u}_{ij} r_{ij}\right) + (\mathbf{v}_i - \mathbf{c}_{ij})^\top (\mathbf{v}_i - \mathbf{c}_{ij}), \\ B &= \left(\mathbf{c}_l^0\right)^\top \left(\mathbf{c}_l^0\right) + (\mathbf{c}_l - \mathbf{t}_j)^\top (\mathbf{c}_l - \mathbf{t}_j). \end{aligned} \tag{13}$$

This least-square rigid motion problem is equivalent to maximizing the following energy F_j:

$$\begin{aligned} &F_j \\ &= \sum_{\mathbf{v}_i \in \mathcal{C}_j} \left(\left(\mathbf{R}_j \mathbf{u}_{ij} r_{ij}\right)^\top (\mathbf{v}_i - \mathbf{c}_{ij})\right) + \omega \sum_{\mathbf{m}_l \in \mathcal{V}_j} \left(\left(\mathbf{R}_j \mathbf{c}_l^0\right)^\top (\mathbf{c}_l - \mathbf{t}_j)\right) \\ &= trace\left(\sum_{\mathbf{v}_i \in \mathcal{C}_j} \left((\mathbf{u}_{ij} r_{ij})^\top \mathbf{R}_j^\top (\mathbf{v}_i - \mathbf{c}_{ij})\right)\right) \\ &+ trace\left(\omega \sum_{\mathbf{m}_l \in \mathcal{V}_j} \left((\mathbf{c}_l^0)^\top \mathbf{R}_j^\top (\mathbf{c}_l - \mathbf{t}_j)\right)\right) \\ &= trace\left(\mathbf{R}_j^\top \sum_{\mathbf{v}_i \in \mathcal{C}_j} \left((\mathbf{v}_i - \mathbf{c}_{ij})(\mathbf{u}_{ij} r_{ij})^\top\right)\right) \\ &+ trace\left(\mathbf{R}_j^\top \left(\omega \sum_{\mathbf{m}_l \in \mathcal{V}_j} \left((\mathbf{c}_l - \mathbf{t}_j)(\mathbf{c}_l^0)^\top\right)\right)\right). \end{aligned} \tag{14}$$

This maximization problem could be solved by decompositing the following matrix:

$$\mathbf{D}_j = \sum_{\mathbf{v}_i \in \mathcal{C}_j} (\mathbf{v}_i - \mathbf{c}_{ij})(\mathbf{u}_{ij} r_{ij})^\top + \omega \sum_{\mathbf{m}_l \in \mathcal{V}_j} (\mathbf{c}_l - \mathbf{t}_j)(\mathbf{c}_l^0)^\top, \tag{15}$$

using Singular Value Decomposition (SVD): $\mathbf{D}_j = \mathbf{U}_j \mathbf{S}_j \mathbf{V}_j^\top$. Then the optimal rotation $\mathbf{R}_j$ can be obtained by:

$$\mathbf{R}_j = \mathbf{U}_j \mathbf{V}_j^\top. \tag{16}$$

Note that we need to check whether $\mathbf{R}_j = \mathbf{U}_j \mathbf{V}_j^\top$ is a rotation. When $\det\left(\mathbf{U}_j \mathbf{V}_j^\top\right) = -1$, it contains reflection, and we reformulate $\mathbf{R}_j$ as:

$$\mathbf{R}_j = \mathbf{U}_j \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \mathbf{V}_j^\top. \tag{17}$$

3. Optimization of Medial Spheres

In this section, we will discuss on the optimization of medial mesh with N medial spheres by minimizing the following expanded total energy E in both optimization stage:

$$E(\{\mathbf{m}_l\}) = \sum_l \sum_{j \in \mathbb{N}(l)} (Q_j + \omega W_j), \tag{18}$$

where $\mathbb{N}(l)$ restores the neighboring medial primitives of medial sphere $\mathbf{m}_l = \{\mathbf{c}_l, r_l\}$, $\mathbf{c}_l$ is the center position of medial vertex $\mathbf{m}_l$ at deformed frame t, and denote $\mathbf{c}_l^0$ the center position of it at reference frame.

Denote $\mathbf{X} \in \mathbb{R}^n$ as the unknowns for both optimization stage. To solve the minimization problem, we set the gradient w.r.t. the unknowns $\mathbf{X}$ to zero. Doing so for the centers $\mathbf{X}$ yields:

$$\mathbf{0} = \frac{\partial E}{\partial \mathbf{X}} = \sum_l \sum_{j \in \mathbb{N}(l)} \left(\frac{\partial Q_j}{\partial \mathbf{X}} + \omega \frac{\partial W_j}{\partial \mathbf{X}}\right). \tag{19}$$

In the first optimization stage, we fix the radii of medial spheres and set $\mathbf{X} = \left(\mathbf{c}_1^\top, \mathbf{c}_2^\top, ..., \mathbf{c}_\mathrm{N}^\top\right)^\top \in \mathbb{R}^{3\mathrm{N}}$, then Eq. (19) becomes:

$$\mathbf{0} = \left(\left(\frac{\partial E}{\mathbf{c}_1}\right)^\top, \left(\frac{\partial E}{\mathbf{c}_2}\right)^\top, ..., \left(\frac{\partial E}{\mathbf{c}_\mathrm{N}}\right)^\top\right)^\top \tag{20}$$

with

$$\begin{aligned} \frac{\partial Q_j}{\partial \mathbf{c}_l} = \sum_{\mathbf{v}_i \in \mathcal{C}_j} 2\alpha_{ijl} \Bigg(\mathbf{R}_j \mathbf{u}_{ij} &\sum_{\mathbf{m}_k \in \mathcal{V}_j} \left(\alpha_{ijk} r_k\right) \\ -\mathbf{v}_i + &\sum_{\mathbf{m}_k \in \mathcal{V}_j} \left(\alpha_{ijk} \mathbf{c}_k\right)\Bigg) \end{aligned} \tag{21}$$

and

$$\frac{\partial W_j}{\partial \mathbf{c}_l} = -2\left(\mathbf{R}_j \mathbf{c}_l^0 + \mathbf{t}_j\right) + 2\mathbf{c}_l, \tag{22}$$

where $\mathcal{C}_j$ and $\mathcal{V}_j$ restores the correspondences and medial spheres of medial primitive $\mathbf{P}_j$ respectively. By replacing $\frac{\partial Q_j}{\partial \mathbf{c}_l}$ and $\frac{\partial W_j}{\partial \mathbf{c}_l}$ into Eq. (20), it becomes a linear problem of $\mathbf{X}$, and Eq. (20) could be rewritten as:

$$\mathbf{A}_{3\mathrm{N}\times 3\mathrm{N}} \mathbf{X}_{3\mathrm{N}\times 1} = \mathbf{b}_{3\mathrm{N}\times 1}. \tag{23}$$

In all our experiments, $\mathbf{A}$ is invertible, and $\mathbf{X}$ could be easily solved by: $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$.

In the second optimization stage, we optimize both the centers and the radii of medial spheres and set

$\mathbf{X} = \left(\mathbf{c}_1^\top, r_1, \mathbf{c}_2^\top, r2, ..., \mathbf{c}_\mathrm{N}^\top, r_\mathrm{N}\right)^\top \in \mathbb{R}^{4\mathrm{N}}$, then Eq. (19) becomes:

$$\mathbf{0} = \left(\left(\frac{\partial E}{\mathbf{c}_1}\right)^\top, \frac{\partial E}{r_1}, \left(\frac{\partial E}{\mathbf{c}_2}\right)^\top, \frac{\partial E}{r_2}, ..., \left(\frac{\partial E}{\mathbf{c}_\mathrm{N}}\right)^\top, \frac{\partial E}{r_\mathrm{N}}\right)^\top \tag{24}$$

with

$$\begin{aligned}\frac{\partial E}{\partial r_l} = \sum_{j \in \mathbb{N}(l)} \sum_{\mathbf{v}_i \in \mathcal{C}_j} 2\alpha_{ijl} \Bigg(&(\mathbf{R}_j \mathbf{u}_{ij})^\top (\mathbf{R}_j \mathbf{u}_{ij}) \Big(\sum_{\mathbf{m}_k \in \mathcal{V}_j} \alpha_{ijk} r_k \Big) + \\ &\Big(-\mathbf{v}_i + \sum_{\mathbf{m}_k \in \mathcal{V}_j} \alpha_{ijk} \mathbf{c}_k \Big)^\top (\mathbf{R}_j \mathbf{u}_{ij}) \Bigg).\end{aligned} \tag{25}$$

Similar to the first optimization stage, by replacing $\frac{\partial Q_j}{\partial \mathbf{c}_l}$, $\frac{\partial W_j}{\partial \mathbf{c}_l}$ and $\frac{\partial E}{\partial r_l}$ into Eq. (24), it becomes a linear problem of $\mathbf{X}$, and Eq. (24) could be rewritten as:

$$\mathbf{A}_{4\mathrm{N} \times 4\mathrm{N}} \mathbf{X}_{4\mathrm{N} \times 1} = \mathbf{b}_{4\mathrm{N} \times 1}. \tag{26}$$

Note that for any surface point $\mathbf{v}_i \in \mathcal{C}_j$, when medial primitive $\mathbf{P}_j$ is a neighboring medial primitive of sphere, say $\mathbf{m}_l$, and the corresponding barycentric coordinate α_{ijl} satisfies: $\alpha_{ijl} \equiv 0$, then the matrix $\mathbf{A}$ isn't invertible because the gradient $\frac{\partial E}{\partial \mathbf{X}}$ is linearly independent with the radius r_l of $\mathbf{m}_l$. For solving this problem, we remove the unknown r_l from $\mathbf{X}$, and keep it unchanged in the corresponding iteration, i.e., $\mathbf{X} = \left(\mathbf{c}_1^\top, r_1, \cdots, \mathbf{c}_l^\top, \mathbf{c}_{l+1}^\top, r_{l+1}, \cdots, \mathbf{c}_\mathrm{N}^\top, r_\mathrm{N}\right)^\top \in \mathbb{R}^{4\mathrm{N}-1}$. Assume that there are $\mathbf{K}$ spheres satisfy the condition, then $\mathbf{X} \in \mathbb{R}^{4\mathrm{N}-K}$. Similar to the first stage, $\mathbf{X}$ could be solved by: $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$.

4. Selection of Reference Frame

Figure 2 shows the comparison of reconstruction errors on the Horse-gallop sequence when we use frame 0 and frame 18 as the reference frame. Some of the frames have larger errors when using frame 18 than using frame 0, while the errors of the other frames are smaller. And the domains with large errors are basically the domains with severe volume change while the consistent medial mesh is not able to capture the large deformation. Besides, when we use frame 18 as the reference frame, the reconstruction error of frame 18 is larger than many other frames while it is the closest mesh of the reference frame in the sequence. This means that the reconstruction errors are not directly affected by the reference surface but the reference medial mesh.

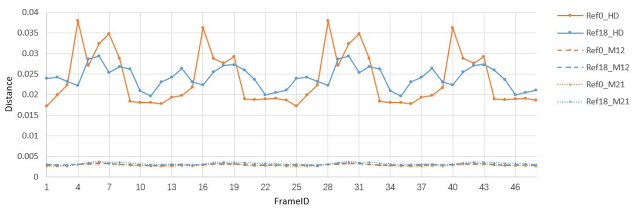

Figure 2: *Distances of selecting different reference frame.*